

Intelligent Systems, Control and Automation:
Science and Engineering

Danwei Wang
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Satellite Formation Flying

Relative Dynamics, Formation Design,
Fuel Optimal Maneuvers and Formation
Maintenance

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This work is dedicated to

*My wife, Ming Luo, and Our children, Anmin,
Andai and Anyu*

—Danwei Wang

My wife, Zhe Jing, and My son, Lihang

—Baolin Wu

*My wife, Poh Ling, and Our children
Clarinda, Emelia and Sherwin*

—Eng Kee Poh

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Chapter 1

Introduction

1.1 Background

1.1.1 Motivations

Autonomous formation flying of multiple small satellites to replace a single large satellite will be an enabling technology for a number of future defense- and science-based space missions. Formation flying is defined as a set of more than one satellite whose dynamic states are coupled through a common control law. In particular, at least one member of the set must (1) track a desired state relative to another member, and (2) the tracking control law must make use of the state of at least one of other members. The second point is critical. For example, even though specific relative positions are actively maintained, the Global Positioning System (GPS) satellites constitute a constellation but not formation flying since orbit correction control law of GPS satellites only makes use of their own position and velocity. Though there are potential limitations, the benefits of using multiple satellites are obvious. As compared to a conventional single large satellite, satellite formation can increase the reliability and redundancy of the whole system, reduce the cost of launching and maintenance, greatly widen the surveillance area, and add more flexibility into the mission design. For instance, a ground observing sensor can be carried by several satellites flying in a specified formation to increase aperture size rather than construct a large and much more expensive single satellite. With a single satellite, the whole mission will be aborted in the event of satellite failure. However, a failed satellite in a formation can be mitigated in the sense that mission can be reorganized using remaining member satellites in the formation. In addition, formation flying technology enables us to easily add more satellites into the mission or upgrade any satellite.

1.1.2 Applications

Potential applications of satellite formation flying include synthetic apertures for surveillance and interferometry missions, and for field measurements and atmospheric survey missions, as well as on-orbit satellite inspection, maintenance, and recovery. There are a number of missions which benefit from satellite formation in either low Earth orbit (LEO) or deep space. For example, TechSat21 (Martin et al. 2001) and Orion/Emerald (Busse et al. 2001), though never launched, paved the way for cooperative satellite operations. Several missions have been deployed, including EO-1/LandSat7 (Ungar et al. 2003), GRACE (Tapley et al. 2004), CLUSTER II/Phoenix (Escoubet et al. 2012), FORMOSAT-3/COSMIC (Wu et al. 2005), PRISMA (Bodin et al. 2012), and TanDEM-X/TerraSAR-X (Buckreuss et al. 2015). The CLUSTER, PRISMA, and TanDEM-X/TerraSAR-X missions are shown in Figs. 1.1, 1.2, and 1.3, respectively. In particular, Magnetospheric Multiscale Mission (MMS) (Fuselier et al. 2014) was launched in March 12, 2015. MMS is a National Aeronautics and Space Administration (NASA) unmanned space mission, to study the Earth's magnetosphere using four identical spacecraft flying in a tetrahedral formation. It is designed to gather information about the

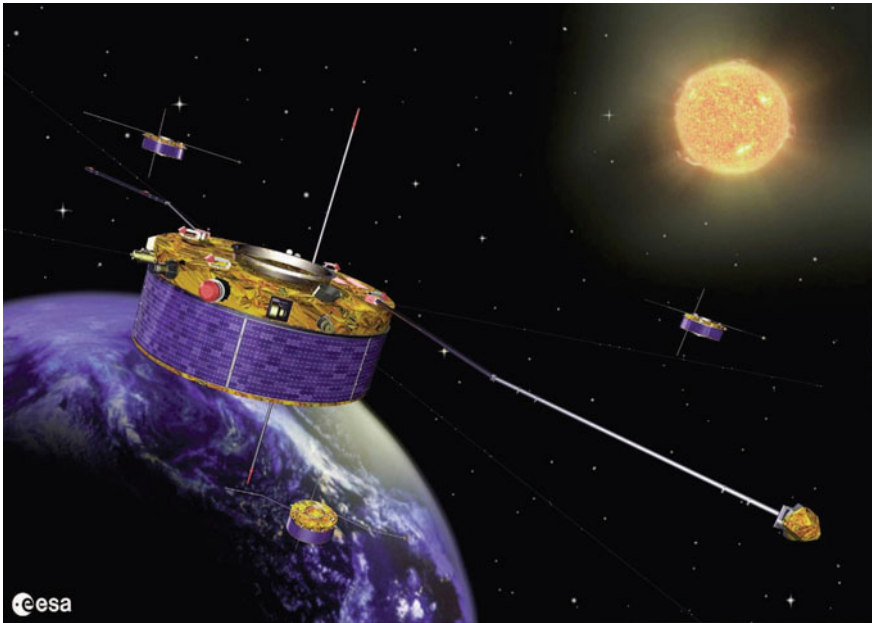


Fig. 1.1 Cluster formation. European Space Agency (ESA)'s cluster is currently investigating the Earth's magnetic environment and its interaction with the solar wind in three dimensions. Science output from cluster greatly advances our knowledge of space plasma physics, space weather, and the Sun–Earth connection and has been a key in improving the modeling of the magnetosphere and understanding its various physical processes (<http://sci.esa.int/cluster>)

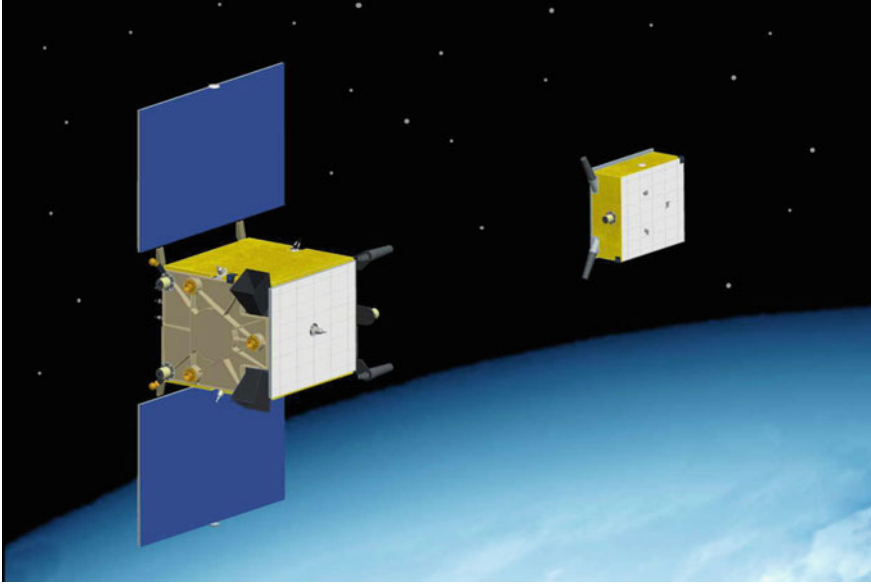


Fig. 1.2 PRISMA provides a technology demonstration mission for the in-flight validation of sensor technologies and guidance/navigation strategies for spacecraft formation flying and rendezvous. PRISMA originates from an initiative of the Swedish National Space Board (SNSB) and the Swedish Space Corporation (SSC) and provides a precursor mission for critical technologies related to advanced formation flying and in-orbit Servicing (<http://www.weblab.dlr.de/rbrt/GpsNav/Prisma/Prisma.html>)

microphysics of magnetic reconnection, energetic particle acceleration, and turbulence, processes that occur in many astrophysical plasmas. All of the missions have prescribed relative motion and attitude requirements, which are designed to meet specific objectives.

1.1.3 Challenges

The advantages from using satellite formations come at a cost of increased complexity and challenges, including high-precision relative navigation (Montenbruck et al. 2011), distributed communication (Smith and Hadaegh 2007), fault detection (Lee et al. 2014; Wu et al. 2014), stable formation design (Roscoe et al. 2013), trajectory optimization and control (Wu et al. 2009, 2016), and attitude synchronization (Wu et al. 2011, 2013). Specifically, the trajectory optimization and control problem can be decomposed into two tasks: (i) Formation keeping to maintain the satellites in a stable formation to within specified precision against various orbital perturbations and (ii) Formation maneuver to guide and execute control command to reconfigure from existing satellite formation to another stable formation.

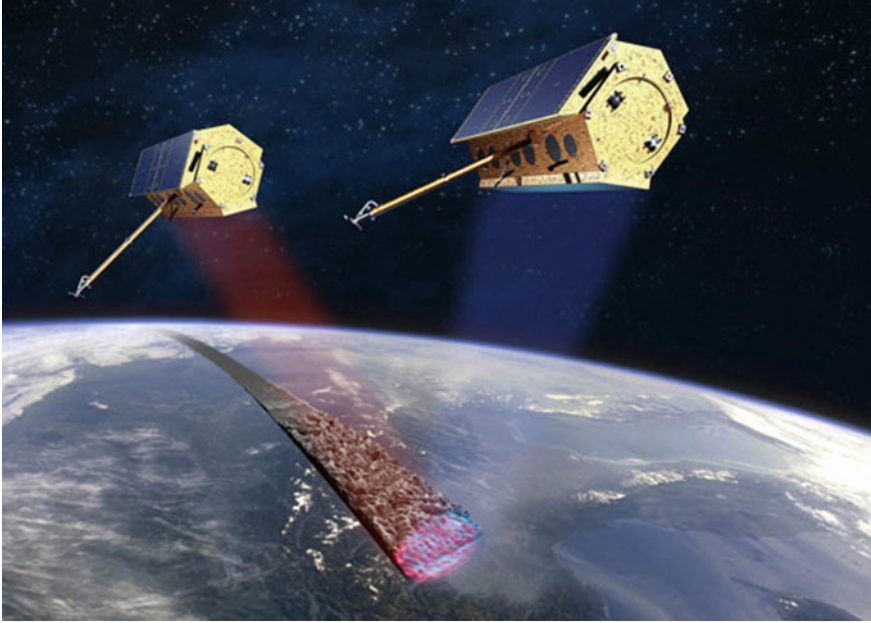


Fig. 1.3 The primary objective of TanDEM-X/TerraSAR-X mission is the generation of a consistent global Digital Elevation Model (DEM) with an unprecedented accuracy. In the mission, two identical Earth observation satellites using synthetic aperture radar are in a closely controlled formation with typical distances between 250 and 500 m. The mission was implemented in a Public–Private Partnership between the German Aerospace centre (DLR) and EADS Astrium and was launched in June 2010 (<https://en.wikipedia.org/wiki/TanDEM-X>)

Attitude synchronization, to maintain accurate relative orientation between satellites, is also critical in many satellite formation missions. For instance, in interferometry application, the relative orientation between satellites in a formation is required to be maintained precisely during formation maneuvers. In inter-satellite laser communication operation, the participating satellites are also required to maintain accurate relative attitude throughout the communication process.

1.2 Objectives of This Book

The major objectives of this book are to describe the developments of relative motion dynamics, formation design methods, optimal formation maneuver and keeping methods, and attitude synchronization controllers for satellite formation missions.

- Firstly, this book provides detailed high fidelity nonlinear and linear relative motion dynamics considering J_2 perturbation and eccentricity for satellite formation design.
- Secondly, this book designs passive and periodic satellite formations considering J_2 perturbation and eccentricity.
- Thirdly, this book proposes fuel-optimal trajectory optimization methods for satellite formation maneuver using a low-thrust propulsion system.
- Fourthly, this book examines fuel-optimal formation keeping methods in both leader-follower and decentralized frame.
- Lastly, this book presents decentralized control laws for attitude synchronization in the presence of model uncertainties and external disturbances for both undirected and directed inter-satellite communication topologies.

1.3 Preview of Chapters

This book is organized into 7 chapters. The details of each chapter are as follows:

- Chapter 2: An exact J_2 nonlinear model of satellite relative motion is developed based on the Lagrangian mechanics. Next, with the aid of Gegenbauer polynomials, the nonlinear model is linearized to a complete J_2 linear relative model. Furthermore, by means of eliminating the second order J_2 effect, the linear model is further approximated to a first order J_2 linear relative model. Finally, a simulation method with a modeling error index is introduced for comparing and evaluating various relative motion dynamics in the literature.
- Chapter 3: Passive and periodic relative trajectories are a practical option for long-term satellite formation flying. This chapter investigates periodic and quasi-periodic satellite relative orbits at critical inclination. Also, periodic and stable in-plane formations are discussed. In particular, a special in-plane formation named as Almost Constant Separation Formation is achieved by suitably designing the differential secular elements of the argument of perigee and the mean anomaly.
- Chapter 4: A fuel-optimal nonlinear trajectory optimization approach using Legendre pseudospectral method is proposed for low-thrust satellite formation maneuver. Accurate and collision-free maneuver trajectories are obtained, and bounded desired final satellite formations are ensured.
- Chapter 5: A real-time optimal control law in a leader-follower frame is developed for satellite formation keeping in elliptic orbits. The proposed control law can prioritize between fuel consumption and control accuracy subject to the control saturation constraints. Next, decentralized control using only local relative measurements is proposed for satellite formation keeping in mildly elliptic orbits. The proposed decentralized formation control architecture is completely decentralized, and scalable.

- Chapter 6: Two decentralized coordinated attitude control laws are developed for attitude synchronization with undirected inter-satellite communication topology. Firstly, an adaptive sliding mode control law is developed against inertia matrix uncertainties and external disturbances. In the event that angular velocity measurement is unavailable, a novel coordinated attitude control law is subsequently proposed.
- Chapter 7: Two decentralized adaptive cooperative attitude control laws are developed for directed inter-satellite communication topology. A robust adaptive control law is proposed to handle inertia matrix uncertainties and external disturbances. In addition, an adaptive backstepping control law which takes into account communication delay is proposed.

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Chapter 2

Dynamic Models of Satellite Relative Motion Around an Oblate Earth

Accurate dynamic model of relative motion is basic and critical to the study of satellite formation flying. Hence, accurate nonlinear and linear dynamic models of satellite relative motion considering J_2 perturbation are derived in this chapter. Firstly, an exact J_2 nonlinear model of satellite relative motion is developed based on the Lagrangian mechanics. Subsequently, with the aid of Gegenbauer polynomials, the nonlinear model is linearized to a complete J_2 linear relative model. Finally, by means of eliminating the second-order J_2 effect, the linear model is further approximated to a first-order J_2 linear relative model. Simulation results show that the exact J_2 nonlinear model produces exact results, and the first-order J_2 linear model also performs well under conditions that the inter-satellite distance is small and time duration is short.

Many relative dynamic models have been derived in the literature under different assumptions and using different methodologies. A comparison study is necessary to select an appropriate model for a specific mission and determine what kind of perturbation should be considered for specific applications. Thus, in this chapter, a simulation method with a modeling error index is also introduced for comparing and evaluating various existing models for relative motion of satellites flying in formation. The comparison results show that, when the Earth aspherical gravity and the air drag are present, the accuracy of some models is affected adversely by eccentricity, semimajor axis, inclination, and formation size. The numerical results provide valuable information for formation design.

2.1 Introduction

The autonomous formation flying of multiple small satellites to replace a single large satellite will be an enabling technology for many future space missions. Potential applications include surveillance missions, field measurement missions, and atmospheric survey missions. With the desire to maneuver or keep a long-term flying

satellite formation comes the need to predict accurate relative position and velocity between satellites. Many studies on relative motion of satellites flying in motion have been reported in the literature. Researchers initially use a set of linearized differential equations, i.e., Hill's equations (Hill 1878), which is also known as Clohessy-Wiltshire (CW) equations (Clohessy and Wiltshire 1960), to describe the relative motion of two satellites in near-circular orbits. Though CW equations had been successfully used in rendezvous scenario, its model error accumulates over time such that its solution becomes erroneous and unacceptable for long duration formation flying.

There has been active research interest to derive a simple and accurate dynamic model for formation design, guidance and control. Many dynamic models have been developed to cater various applications. Generally speaking, these dynamic models can be classified into three categories. The first category is the direct ordinary differential equation (ODE) models, which are mostly extensions or modifications of CW equations (Tschauner and Hempel 1965; Kechichian 1998; Schweighart and Sedwick 2002, 2005; Tillerson and How 2002; Vadali et al. 2001; Ross 2003; Roberts and Roberts 2004; Pluym and Damaren 2006; Inalhan et al. 2002; Gurfil 2005; Xu and Wang 2008; Morgan et al. 2012). As these models are in the form of differential equations, they have significant applications in controller design. The second category is the indirect models, which are usually expressed in differences of orbit elements (Schaub and Alfriend 2001; Schaub 2002, 2004). It is easier to use the second category of models to design the satellite formation because they describe the formation geometry directly. The third category is the solution-based models, which are usually in the form of state transition matrix (STM) (Gim and Alfriend 2005; Sengupta et al. 2007; Palmer and Imre 2007; Lee et al. 2007). Using STM, though it is very complicated, one can directly generate the satellite relative motion.

For the direct ODE models, CW equations were first extended to unperturbed relative motion that takes into account of eccentricity and/or nonlinearity. Tschauner and Hempel (1965) solved the satellite relative motion in elliptical orbits. Analytical solution to their model can be derived both in true anomaly and time domain (Tschauner and Hempel 1965; Tillerson and How 2002). The second order J_2 effect is the dominant perturbation for satellite formation missions. As such, a J_2 dynamic model of satellite relative motion in terms of differential equations in the local vertical local horizontal (LVLH) coordinate is significantly useful for the study of satellite formation flying in low Earth orbits. In fact, different J_2 dynamic equations have been developed. A noteworthy work was reported by Kechichian (1998), in which an exact nonlinear relative model that includes both the J_2 perturbation and the air drag was developed. Kechichian applied Newtonian mechanics and applied vector calculus to derive the relative dynamics. His result is very complex and represented equivalently by 12 first-order differential equations. In particular, some components of J_2 acceleration are not explicitly expressed in the relative model, and instead, a tedious algorithm is provided to calculate these J_2 acceleration components.

Some recent attempts have been done on the development of J_2 linear dynamics. Schweighart and Sedwick (2002, 2005) developed a hybrid J_2 linear model in near-circular orbits using averaged J_2 acceleration. Equations of this model on the

in-plane motion are linear time invariant (LTI) and somewhat similar to CW model (Clohessy and Wiltshire 1960), while equations to describe the cross-track motion are linear time varying (LTV). To simplify the cross-track dynamics, Tillerson and How (2002) combined the cross-track dynamics proposed by Vadali et al. (2001) with the in-plane dynamics of Schweighart and Sedwick. Some other results are reported (Ross 2003; Roberts and Roberts 2004; Pluym and Damaren 2006). A common assumption of those works is that the reference satellite flies in an unperturbed Keplerian orbit. As a result, the accuracy and the applicability of these models are limited.

In this chapter, similar to Kechichian's work, the satellite relative dynamics is studied based on the perturbed reference orbit, which is accurately described by a set of differential equations (Xu and Wang 2008b). A modification to the Kechichian's method is that the reference orbit is expressed simpler in terms of Reference Satellite Variables (RSV). Different from Kechichian's techniques, Lagrangian mechanics is used to derive the satellite relative dynamics. Consequently, an exact J_2 nonlinear relative model is obtained, which is independent of the right ascension of ascending node such that the satellite relative motion is completely expressed by 11 simple first-order differential equations. Furthermore, the complete J_2 linear relative model and the first-order J_2 linear relative model are derived by removing the nonlinear effect and the second-order J_2 effect (Xu and Wang 2008b). By means of numerical study, it is found that the nonlinear effect and the second-order J_2 effects introduce errors in opposite directions, so that they counteract each other somewhat. As a result, the first-order J_2 linear relative model is suitable for short-duration formation maneuver and keeping.

2.2 Nonlinear Dynamic Model of Relative Motion

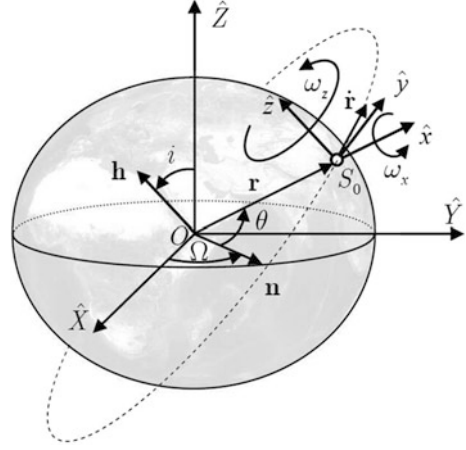
2.2.1 J_2 Reference Satellite Dynamics in LVLH Frame

In the study, one satellite, or a virtual satellite, is taken as reference satellite and others as member satellites. Without loss of generality, a 2-satellite system is considered, i.e., a free-flying reference satellite S_0 (without control force) and a controlled member satellite S_j (with control force). This subsection is devoted to explicitly establish the J_2 dynamics of the single reference satellite S_0 in the local rotating frame.

2.2.1.1 Properties of LVLH Frame

Two Cartesian coordinates are used in this study. As shown in Fig. 2.1, Earth-centered inertial (ECI) coordinate is spanned by unit vectors $(\hat{\mathbf{X}}, \hat{\mathbf{Y}}, \hat{\mathbf{Z}})$. LVLH coordinate $(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}})$ is attached on reference satellite S_0 . Vectors $\mathbf{r}$ and $\dot{\mathbf{r}}$ denote the position and the velocity of satellite S_0 , respectively. Hence, the vector

Fig. 2.1 ECI and LVLH coordinates



of angular momentum per unit mass is defined by $\mathbf{h} = \mathbf{r} \times \dot{\mathbf{r}}$. Furthermore, $r = \|\mathbf{r}\|$ and $h = \|\mathbf{h}\|$ denote the geocentric distance and the magnitude of angular momentum of satellite S_0 , where $\|\cdot\|$ denotes Euclidean norm. Then, the LVLH coordinate is spanned by unit vectors

$$\hat{\mathbf{x}} = \frac{\mathbf{r}}{r} \quad \hat{\mathbf{y}} = \hat{\mathbf{z}} \times \hat{\mathbf{x}} \quad \hat{\mathbf{z}} = \frac{\mathbf{h}}{h} \quad (2.1)$$

Vectors in LVLH coordinate and ECI coordinate can be transformed (Battin 1999) to each other by the rotation matrix R as

$$\begin{bmatrix} \hat{\mathbf{X}} & \hat{\mathbf{Y}} & \hat{\mathbf{Z}} \end{bmatrix}^T = R \begin{bmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \end{bmatrix}^T \quad (2.2)$$

with

$$R = \begin{bmatrix} c_\theta c_\Omega - s_\theta c_i s_\Omega & -s_\theta c_\Omega - c_\theta c_i s_\Omega & s_i s_\Omega \\ c_\theta s_\Omega + s_\theta c_i c_\Omega & -s_\theta s_\Omega + c_\theta c_i c_\Omega & -s_i c_\Omega \\ s_\theta s_i & c_\theta s_i & c_i \end{bmatrix} \quad (2.3)$$

where $s_\circ = \sin(\circ)$, $c_\circ = \cos(\circ)$, i is the inclination, θ is the true latitude, and Ω is the right ascension of ascending node.

The angular velocity of rotating LVLH frame is

$$\boldsymbol{\omega} = \omega_x \hat{\mathbf{x}} + \omega_z \hat{\mathbf{z}} \quad (2.4)$$

Note that the component around y -axis is zero, i.e., $\omega_y = 0$ (Kechichian 1998). The component ω_z is referred as the orbital rate, and the component ω_x denotes the steering rate of the orbital plane. Angular velocity $\boldsymbol{\omega}$ can be expressed by Eulerian angles (Ω, i, θ) , and its three components are (Kechichian 1998; Breakwell 1974):

$$\omega_x = \dot{c}_\theta + \dot{\Omega}s_\theta s_i \quad (2.5)$$

$$\omega_y = -\dot{s}_\theta + \dot{\Omega}c_\theta s_i = 0 \quad (2.6)$$

$$\omega_z = \dot{\theta} + \dot{\Omega}c_i \quad (2.7)$$

Using Eq. (2.4), velocities of unit vectors of LVLH frame are expressed as follows:

$$\dot{\hat{\mathbf{x}}} = \boldsymbol{\omega} \times \hat{\mathbf{x}} = \omega_z \hat{\mathbf{y}} \quad \dot{\hat{\mathbf{y}}} = \boldsymbol{\omega} \times \hat{\mathbf{y}} = \omega_x \hat{\mathbf{z}} - \omega_z \hat{\mathbf{x}} \quad \dot{\hat{\mathbf{z}}} = \boldsymbol{\omega} \times \hat{\mathbf{z}} = -\omega_x \hat{\mathbf{y}} \quad (2.8)$$

Then, the velocity of satellite S_0 is computed to

$$\dot{\mathbf{r}} = \frac{d}{dt}(r\hat{\mathbf{x}}) = \dot{r}\hat{\mathbf{x}} + r\omega_z\hat{\mathbf{y}} \quad (2.9)$$

Using Eq. (2.9), the angular momentum can be expressed as

$$\mathbf{h} = \mathbf{r} \times \dot{\mathbf{r}} = r^2\omega_z\hat{\mathbf{z}} \quad (2.10)$$

Comparing Eq. (2.10) with the last equation in (2.1), it is concluded that the magnitude of angular momentum is

$$h = \omega_z r^2 \quad (2.11)$$

Taking time derivative to Eq. (2.11), the rate of angular momentum is obtained as

$$\dot{h} = 2\omega_z r \dot{r} + \dot{\omega}_z r^2 \quad (2.12)$$

2.2.1.2 J_2 Dynamics of a Satellite in LVLH Frame

The dynamics of reference satellite S_0 in LVLH frame considering both the spherical gravitational potential and the oblate (J_2) gravitational potential is derived in this section. The governing equation is

$$\ddot{\mathbf{r}} = -\nabla U \quad (2.13)$$

with

$$U = -\frac{\mu}{r} - \frac{3J_2\mu R_e^2}{2r^3} \left(\frac{1}{3} - s_\phi^2 \right) \quad (2.14)$$

where U is the gravitational potential energy of satellite. ∇ denotes the vector differential operator. μ is the Earth gravitational constant. J_2 is the second zonal harmonic coefficient of the Earth. R_e is the Earth equatorial radius, and ϕ is the geocentric latitude of satellite S_0 .

Using Eqs. (2.8) and (2.9), the left side of Eq. (2.13) is calculated as

$$\ddot{\mathbf{r}} = (\ddot{r} - \omega_z^2 r)\hat{\mathbf{x}} + (\dot{\omega}_z r + 2\omega_z \dot{r})\hat{\mathbf{y}} + (\omega_x \omega_z r)\hat{\mathbf{z}} \quad (2.15)$$

Using Eqs. (2.11) and (2.12), ω_z and $\dot{\omega}_z$ in Eq. (2.15) can be replaced by h and $\dot{h}$, and then, Eq. (2.15) is converted to

$$\ddot{\mathbf{r}} = \left(\ddot{r} - \frac{h^2}{r^3} \right) \hat{\mathbf{x}} + \frac{\dot{h}}{r} \hat{\mathbf{y}} + \frac{\omega_x h}{r} \hat{\mathbf{z}} \quad (2.16)$$

Next, the gradient of U in Eq. (2.14) is computed in LVLH frame to be

$$\nabla U = \frac{\mu}{r^2} \hat{\mathbf{x}} + \frac{k_{J2}}{r^4} (1 - 3s_i^2 s_\theta^2) \hat{\mathbf{x}} + \frac{k_{J2} s_i^2 s_{2\theta}}{r^4} \hat{\mathbf{y}} + \frac{k_{J2} s_{2i} s_\theta}{r^4} \hat{\mathbf{z}} \quad (2.17)$$

where the constant k_{J2} is defined as

$$k_{J2} = \frac{3J_2 \mu R_e^2}{2} \quad (2.18)$$

Substituting Eqs. (2.16) and (2.17) into Eq. (2.13), the below three equations are obtained

$$\omega_x = -\frac{k_{J2} s_{2i} s_\theta}{hr^3} \quad (2.19)$$

$$\ddot{r} = -\frac{\mu}{r^2} + \frac{h^2}{r^3} - \frac{k_{J2}}{r^4} (1 - 3s_i^2 s_\theta^2) \quad (2.20)$$

$$\dot{h} = -\frac{k_{J2} s_i^2 s_{2\theta}}{r^3} \quad (2.21)$$

Replacing ω_x in Eq. (2.5) by Eq. (2.19), differential terms $(\dot{\Omega}, \dot{i})$ are solved using Eqs. (2.5) and (2.6)

$$\dot{\Omega} = -\frac{2k_{J2} c_i s_\theta^2}{hr^3} \quad (2.22)$$

$$\dot{i} = -\frac{k_{J2} s_{2i} s_{2\theta}}{2hr^3} \quad (2.23)$$

Substituting Eq. (2.22) into Eq. (2.7), and considering Eq. (2.11), $\dot{\theta}$ is also solved

$$\dot{\theta} = \frac{h}{r^2} + \frac{2k_{J2}c_i^2s_{\theta}^2}{hr^3} \quad (2.24)$$

The results established are presented in the following theorem.

Theorem 2.1 *Considering spherical gravity and J_2 gravity of the Earth, the motion of the reference satellite S_0 can be described by a set of differential equations as in Eqs. (2.20), (2.21), (2.23) and (2.24) in terms of Reference Satellite Variables (RSV) $(\dot{r}, r, h, \Omega, i, \theta)$.*

Remark 2.1 It is noticed that five variables $(\dot{r}, r, h, i, \theta)$ are independent of Ω . This is expected because J_2 gravity is independent of the change of Ω .

The rotation rate of LVLH frame can be expressed conveniently in terms of RSV and also independent of Ω . The steering rate ω_x is obtained in Eq. (2.19). The orbital rate ω_z is the direct result of (2.11), i.e.,

$$\omega_z = \frac{h}{r^2} \quad (2.25)$$

Taking time derivative Eqs. (2.19) and (2.25), respectively, and considering Eqs. (2.21), (2.23), and (2.24), the steering acceleration α_x and the orbital acceleration α_z are derived as

$$\alpha_x = \dot{\omega}_x = -\frac{k_{J2}s_{2i}c_{\theta}}{r^5} + \frac{3\dot{r}k_{J2}s_{2i}s_{\theta}}{r^4h} - \frac{8k_{J2}^2s_i^3c_is_{\theta}^2c_{\theta}}{r^6h^2} \quad (2.26)$$

$$\alpha_z = \dot{\omega}_z = -\frac{2h\dot{r}}{r^3} - \frac{k_{J2}s_i^2s_{2\theta}}{r^5} \quad (2.27)$$

2.2.2 Derivation of Exact J_2 Nonlinear Relative Dynamics

2.2.2.1 Lagrangian Formulation of Relative Motion

The Lagrangian formulation is used to develop the relative dynamics of the member satellite S_j . The Lagrangian formulation for satellite relative motion is

$$\frac{d}{dt} \left(\frac{\partial L_j}{\partial \dot{\mathbf{q}}_j} \right) - \frac{\partial L_j}{\partial \mathbf{q}_j} = \mathbf{a}_j \quad (2.28)$$

where $\mathbf{q}_j = [x_j \ y_j \ z_j]^T$ and $\mathbf{a}_j = [a_{jx} \ a_{jy} \ a_{jz}]^T$ are respectively the configurations and the control accelerations of satellite S_j in LVLH coordinate, and L_j is its Lagrangian, which can be further expressed in the form of

$$L_j(\mathbf{q}_0, \dot{\mathbf{q}}_0, \mathbf{q}_j, \dot{\mathbf{q}}_j) = K_j(\mathbf{q}_0, \dot{\mathbf{q}}_0, \mathbf{q}_j, \dot{\mathbf{q}}_j) - U_j(\mathbf{q}_0, \mathbf{q}_j) \quad (2.29)$$

where $\mathbf{q}_0$ are configurations of reference satellite S_0 in ECI coordinate. K_j and U_j are respectively the kinetic and potential energies of the j -th member satellite. Since kinetic energy K_j arises from inertial motion, it depends on the relative motion $(\mathbf{q}_j, \dot{\mathbf{q}}_j)$ of satellite S_j in LVLH coordinate as well as the transport motion $(\mathbf{q}_0, \dot{\mathbf{q}}_0)$ of LVLH frame in ECI coordinate. On the other hand, potential energy U_j is solely due to gravity and thus is independent of velocities. Substituting Eq. (2.29) into Eq. (2.28) yields

$$\frac{d}{dt} \left(\frac{\partial K_j}{\partial \dot{\mathbf{q}}_j} \right) - \frac{\partial K_j}{\partial \mathbf{q}_j} + \frac{\partial U_j}{\partial \mathbf{q}_j} = \mathbf{a}_j \quad (2.30)$$

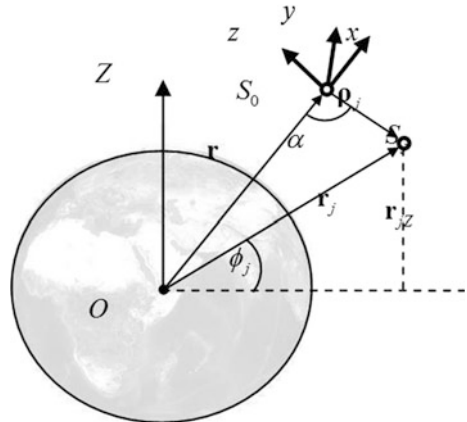
which is the Lagrangian formulation for the relative motion of member satellite S_j in LVLH frame. Next, Eq. (2.30) is used to develop the dynamics of satellite relative motion. The key steps are the precise calculations of kinetic energy K_j and potential energy U_j .

2.2.2.2 Kinetic Energy

As shown in Fig. 2.2, the position of satellite S_j in LVLH frame is

$$\boldsymbol{\rho}_j = x_j \hat{\mathbf{x}} + y_j \hat{\mathbf{y}} + z_j \hat{\mathbf{z}} \quad (2.31)$$

Fig. 2.2 Geometry of the member satellite



Then, in ECI frame, it is

$$\mathbf{r}_j = \mathbf{r} + \boldsymbol{\rho}_j = (x_j + r)\hat{\mathbf{x}} + y_j\hat{\mathbf{y}} + z_j\hat{\mathbf{z}} \quad (2.32)$$

Using the identities in Eq. (2.8), the velocity of satellite S_j in inertial space is calculated by taking time derivative of Eq. (2.32)

$$\dot{\mathbf{r}}_j = v_{jx}\hat{\mathbf{x}} + v_{jy}\hat{\mathbf{y}} + v_{jz}\hat{\mathbf{z}} \quad (2.33)$$

where

$$v_{jx} = \dot{x}_j + \dot{r} - y_j\omega_z \quad v_{jy} = \dot{y}_j + (r + x_j)\omega_z - z_j\omega_x \quad v_{jz} = \dot{z}_j + y_j\omega_x \quad (2.34)$$

Then, the kinetic energy per unit mass of satellite S_j is computed as

$$\begin{aligned} K_j &= \frac{1}{2} \left(v_{jx}^2 + v_{jy}^2 + v_{jz}^2 \right) \\ &= \frac{1}{2} (\dot{x}_j + \dot{r} - y_j\omega_z)^2 + \frac{1}{2} (\dot{y}_j + (r + x_j)\omega_z - z_j\omega_x)^2 + \frac{1}{2} (\dot{z}_j + y_j\omega_x)^2 \end{aligned} \quad (2.35)$$

where time-varying variables $(\dot{r}, r, \omega_x, \omega_z)$ are functions of RSV.

2.2.2.3 Potential Energy

Considering J_2 perturbation, the gravitational potential energy of member satellite S_j is

$$U_j = -\frac{\mu}{r_j} - \frac{k_{J2}}{r_j^3} \left(\frac{1}{3} - s_{\phi_j}^2 \right) \quad (2.36)$$

where ϕ_j and r_j are the geocentric latitude and the geocentric distance of satellite S_j , respectively. From Eq. (2.32), geocentric distance r_j is immediately obtained as

$$r_j = \sqrt{(r + x_j)^2 + y_j^2 + z_j^2} \quad (2.37)$$

From the geometry shown in Fig. 2.2, it is obtained

$$s_{\phi_j} = \frac{r_{jZ}}{r_j} \quad (2.38)$$

where r_{jZ} is the projection of $\mathbf{r}_j$ on Z axis of ECI frame. Since transformation R in Eq. (2.3) is an orthogonal matrix, $\mathbf{r}_j$ in Eq. (2.32) can be transformed to

$$\begin{bmatrix} r_{jX} & r_{jY} & r_{jZ} \end{bmatrix}^T = R \begin{bmatrix} x_j + r & y_j & z_j \end{bmatrix}^T \quad (2.39)$$

where r_{jX} , r_{jY} and r_{jZ} are the respective components of $\mathbf{r}_j$ expressed in ECI frame and

$$r_{jZ} = (r + x_j)s_i s_\theta + y_j s_i c_\theta + z_j c_i \quad (2.40)$$

Substituting Eq. (2.38) into Eq. (2.36), the potential energy of the j -th member satellite is given by

$$U_j = -\frac{\mu}{r_j} - \frac{k_{J2}}{3r_j^3} + \frac{k_{J2}r_{jZ}^2}{r_j^5} \quad (2.41)$$

where r_j and r_{jZ} are expressed in equations Eqs. (2.37) and (2.40).

2.2.2.4 Exact Nonlinear J_2 Relative Dynamics

Having kinetic energy in Eq. (2.35) and potential energy in Eq. (2.41), it is ready to use Lagrangian formulation of Eq. (2.30) to derive the satellite relative dynamics. Substituting Eq. (2.35) into the first two terms of Eq. (2.30), it is obtained

$$\frac{d}{dt} \left(\frac{\partial K_j}{\partial \dot{\mathbf{q}}_j} \right) - \frac{\partial K_j}{\partial \mathbf{q}_j} = \begin{bmatrix} \ddot{x}_j - 2\dot{y}_j\omega_z - x_j\omega_z^2 - y_j\alpha_z + z_j\omega_x\omega_z - r(\omega_z)^2 + (\ddot{r}) \\ \ddot{y}_j + 2\dot{x}_j\omega_z - 2\dot{z}_j\omega_x + x_j\alpha_z - y_j\omega_z^2 - y_j\omega_x^2 - z_j\alpha_x + 2\dot{r}(\omega_z) + r(\alpha_z) \\ \ddot{z}_j + 2\dot{y}_j\omega_x + x_j\omega_x\omega_z + y_j\alpha_x - z_j\omega_x^2 + r(\omega_x\omega_z) \end{bmatrix} \quad (2.42)$$

Next RSV dynamics are applied. The variables in round brackets in Eq. (2.42) are replaced with Eqs. (2.19), (2.20), (2.25), and (2.27). Furthermore, define

$$\zeta = \frac{2k_{J2}s_i s_\theta}{r^4} \quad n^2 = \frac{\mu}{r^3} + \frac{k_{J2}}{r^5} - \frac{5k_{J2}s_i^2 s_\theta^2}{r^5} \quad (2.43)$$

With some manipulations, Eq. (2.42) is converted to

$$\frac{d}{dt} \left(\frac{\partial K_j}{\partial \dot{\mathbf{q}}_j} \right) - \frac{\partial K_j}{\partial \mathbf{q}_j} = \begin{bmatrix} \ddot{x}_j - 2\dot{y}_j\omega_z - x_j\omega_z^2 - y_j\alpha_z + z_j\omega_x\omega_z - rn^2 - \zeta s_i s_\theta \\ \ddot{y}_j + 2\dot{x}_j\omega_z - 2\dot{z}_j\omega_x + x_j\alpha_z - y_j(\omega_z^2 + \omega_x^2) - z_j\alpha_x - \zeta s_i c_\theta \\ \ddot{z}_j + 2\dot{y}_j\omega_x + x_j\omega_x\omega_z + y_j\alpha_x - z_j\omega_x^2 - \zeta c_i \end{bmatrix} \quad (2.44)$$

On the other hand, inserting Eq. (2.41) into the third term in Eq. (2.30), and defining

$$\zeta_j = \frac{2k_{J2}r_{jz}}{r_j^5} \quad n_j^2 = \frac{\mu}{r_j^3} + \frac{k_{J2}}{r_j^5} - \frac{5k_{J2}r_{jz}^2}{r_j^7} \quad (2.45)$$

it is obtained

$$\frac{\partial U_j}{\partial \mathbf{q}_j} = \begin{bmatrix} n_j^2(r + x_j) + \zeta_j s_i s_\theta & n_j^2 y_j + \zeta_j s_i c_\theta & n_j^2 z_j + \zeta_j c_i \end{bmatrix}^T \quad (2.46)$$

Then, substituting Eqs. (2.44) and (2.46) into Eq. (2.30), the exact J_2 nonlinear model of the satellite relative motion is derived to be

$$\begin{aligned} \ddot{x}_j &= 2\dot{y}_j\omega_z - x_j(n_j^2 - \omega_z^2) + y_j\alpha_z - z_j\omega_x\omega_z - (\zeta_j - \zeta)s_i s_\theta - r(n_j^2 - n^2) + a_{jx} \\ \ddot{y}_j &= -2\dot{x}_j\omega_z + 2\dot{z}_j\omega_x - x_j\alpha_z - y_j(n_j^2 - \omega_z^2 - \omega_x^2) + z_j\alpha_x - (\zeta_j - \zeta)s_i c_\theta + a_{jy} \\ \ddot{z}_j &= -2\dot{y}_j\omega_x - x_j\omega_x\omega_z - y_j\alpha_x - z_j(n_j^2 - \omega_x^2) - (\zeta_j - \zeta)c_i + a_{jz} \end{aligned} \quad (2.47)$$

Now, the nonlinear dynamic equations of the satellite relative motion can be presented in the following theorem.

Theorem 2.2 Consider a 2-satellite system of the reference satellite S_0 and the member satellite S_j , as shown in Fig. 2.1. In the presence of spherical gravity and J_2 gravity of the Earth, the relative motion of the satellite S_j in the LVLH coordinate can be described by (2.47). In dynamics (2.47), all time varying parameters $(r, i, \theta, \omega_x, \omega_z, \alpha_x, \alpha_z, \zeta, n^2)$ are functions of RSV, which are given by Eqs. (2.19), (2.23)–(2.27) and (2.43). Variables (ζ_j, n_j^2) are nonlinear terms of configuration (x_j, y_j, z_j) , which are given by Eqs. (2.45), (2.37) and (2.40).

Remark 2.2 Dynamics (2.47) is independent of the motion of right ascension of ascending node Ω . This interesting observation is understandable because only the spherical and J_2 gravities of the Earth are included in the developed dynamics, and both of them are axial symmetric and independent of the motion of Ω . Therefore, the satellite relative motion under J_2 perturbation is actually described by 11 first-order differential equations of $(x_j, y_j, z_j, \dot{x}_j, \dot{y}_j, \dot{z}_j)$ and $(r, \dot{r}, h, i, \theta)$.

Remark 2.3 When relative model (2.47) is applied to study the problems of formation guidance and control, Eqs. (2.20)–(2.24) in RSV dynamics are good candidate to propagate time-varying parameters $(r, \dot{r}, h, i, \theta)$. However, they are not necessary in practice and other techniques can be used to evaluate these parameters, e.g., the technique that combines the advantages of orbit propagator and statistical orbit determination.

2.3 Linearized Dynamic Models of Relative Motion

In dynamics of Eq. (2.47), n_j^2 and ζ_j are nonlinear terms of configurations (x_j, y_j, z_j) , because they both include polynomials of the reciprocal of r_j . The technique of Gegenbauer polynomials is applied in this section to linearize the nonlinear model of Eq. (2.47).

Based on Fig. 2.2 and using the cosine theorem, r_j can be expressed as

$$r_j^2 = r^2 - 2r\rho_j \cos \alpha + \rho_j^2 \quad (2.48)$$

with

$$\cos \alpha = \frac{(-\mathbf{r}) \cdot \mathbf{\rho}_j}{r\rho_j} = -\frac{x_j}{\rho_j} \quad (2.49)$$

Equation (2.48) can be converted to

$$\frac{1}{r_j^{2\lambda}} = \frac{1}{r^{2\lambda}} \left(1 - 2 \cos \alpha \left(\frac{\rho_j}{r} \right) + \left(\frac{\rho_j}{r} \right)^2 \right)^{-\lambda} \quad (2.50)$$

Next, Gegenbauer polynomials (ultraspherical polynomials) is introduced, which are generalizations of the associated Legendre polynomials and expressed as

$$C_n^{(\lambda)}(u) = \frac{\Gamma(\lambda + 1/2)}{\Gamma(2\lambda)} \frac{\Gamma(n + 2\lambda)}{\Gamma(n + \lambda + 1/2)} \frac{(-1)^n}{2^n n!} (1 - u^2)^{-\lambda + 1/2} \frac{d^n}{du^n} (1 - u^2)^{n + \lambda - 1/2} \quad (2.51)$$

where $\Gamma(\circ)$ is the gamma function. The first two terms of Eq. (2.51) are

$$C_0^{(\lambda)}(u) = 1 \quad C_1^{(\lambda)}(u) = 2\lambda u \quad (2.52)$$

The generating function of Gegenbauer polynomials is, for $(\lambda + 1/2) > 0$, $|v| < 1$ and $|u| \leq 1$,

$$(1 - 2uv + v^2)^{-\lambda} = \sum_{n=0}^{\infty} C_n^{(\lambda)}(u) v^n \approx 1 + (2\lambda u)v \quad (2.53)$$

Let

$$u = \cos \alpha \quad v = \frac{\rho_j}{r} \quad (2.54)$$

Equation (2.50) can be expanded in the form of Eq. (2.53) as

$$\frac{1}{r_j^{2\lambda}} \approx \frac{1}{r^{2\lambda}} \left(1 + 2\lambda \left(-\frac{x_j}{r} \right) \left(\frac{\rho_j}{r} \right) \right) = \frac{1}{r^{2\lambda}} \left(1 - \frac{2\lambda x_j}{r} \right) \quad (2.55)$$

For $\lambda = (3/2), (5/2), (7/2)$, Eq. (2.55) gives

$$\frac{1}{r_j^3} \approx \frac{1}{r^3} - \frac{3x_j}{r^4} \quad \frac{1}{r_j^5} \approx \frac{1}{r^5} - \frac{5x_j}{r^6} \quad \frac{1}{r_j^7} \approx \frac{1}{r^7} - \frac{7x_j}{r^8} \quad (2.56)$$

Substituting Eqs. (2.56) and (2.40) into Eq. (2.45), and removing higher-order terms, ζ_j and n_j^2 are linearized to

$$\zeta_j \approx \zeta - \frac{8k_{J2}x_j s_i s_\theta}{r^5} + \frac{2k_{J2}y_j s_i c_\theta}{r^5} + \frac{2k_{J2}z_j c_i}{r^5} \quad (2.57)$$

$$n_j^2 \approx n^2 - \frac{3\mu x_j}{r^4} - \frac{5k_{J2}x_j(1 - 5s_i^2 s_\theta^2)}{r^6} - \frac{5k_{J2}y_j s_i^2 s_{2\theta}}{r^6} - \frac{5k_{J2}z_j s_{2i} s_\theta}{r^6} \quad (2.58)$$

Now, Eqs. (2.57) and (2.58) are substituted into Eq. (2.47). After some operations and removing higher-order terms, the complete J_2 linear model is obtained

$$\begin{aligned} \ddot{x}_j &= 2\dot{y}_j \omega_z + x_j \left(2n^2 + \omega_z^2 + \frac{2k_{J2}}{r^5} (1 - s_i^2 s_\theta^2) \right) \\ &\quad + y_j \left(\alpha_z + \frac{4k_{J2} s_i^2 s_{2\theta}}{r^5} \right) - 5z_j \omega_x \omega_z + a_{jx} \\ \ddot{y}_j &= -2\dot{x}_j \omega_z + 2\dot{z}_j \omega_x + x_j \left(\frac{4k_{J2} s_i^2 s_{2\theta}}{r^5} - \alpha_z \right) \\ &\quad - y_j \left(\frac{2k_{J2} s_i^2 c_\theta^2}{r^5} + n^2 - \omega_z^2 - \omega_x^2 \right) \\ &\quad + z_j \left(\alpha_x - \frac{k_{J2} s_{2i} c_\theta}{r^5} \right) + a_{jy} \\ \ddot{z}_j &= -2\dot{y}_j \omega_x - 5x_j \omega_x \omega_z - y_j \left(\frac{k_{J2} s_{2i} c_\theta}{r^5} + \alpha_x \right) \\ &\quad - z_j \left(n^2 - \omega_x^2 + \frac{2k_{J2} c_i^2}{r^5} \right) + a_{jz} \end{aligned} \quad (2.59)$$

The above equation can be rewritten into the following form

$$\frac{d}{dt} \begin{bmatrix} \dot{x}_j \\ \dot{y}_j \\ \dot{z}_j \end{bmatrix} = A_1(t) \begin{bmatrix} \dot{x}_j \\ \dot{y}_j \\ \dot{z}_j \end{bmatrix} + A_2(t) \begin{bmatrix} x_j \\ y_j \\ z_j \end{bmatrix} + \begin{bmatrix} a_{jx} \\ a_{jy} \\ a_{jz} \end{bmatrix} \quad (2.60)$$

where

$$A_1(t) = \begin{bmatrix} 0 & 2\omega_z & 0 \\ -2\omega_z & 0 & 2\omega_x \\ 0 & -2\omega_x & 0 \end{bmatrix} \quad (2.61)$$

$$A_2(t) = \begin{bmatrix} 2n^2 + \omega_z^2 + \frac{2k_{J2}}{r^5}(1 - s_i^2 s_\theta^2) & \alpha_z + \frac{4k_{J2}s_i^2 s_{2\theta}}{r^5} & -5\omega_x \omega_z \\ \frac{4k_{J2}s_i^2 s_{2\theta}}{r^5} - \alpha_z & \frac{2k_{J2}s_i^2 c_\theta^2}{r^5} + n^2 - \omega_z^2 - \omega_x^2 & \alpha_x - \frac{k_{J2}s_{2i}c_\theta}{r^5} \\ -5x_j \omega_x \omega_z & \frac{k_{J2}s_{2i}c_\theta}{r^5} + \alpha_x & n^2 - \omega_x^2 + \frac{2k_{J2}c_\theta^2}{r^5} \end{bmatrix} \quad (2.62)$$

The above results can be summarized in the following corollary.

Corollary 2.1 *Consider a 2-satellite system of the reference satellite S_0 and the member satellite S_j , as shown in Fig. 2.1. In the presence of spherical gravity and J_2 gravity of the Earth, linear model of the relative motion of the satellite S_j in the LVLH coordinate can be derived as in (2.60)–(2.62) by removal of nonlinear effect.*

Since the nonlinear effect is removed in the linear model of Eq. (2.59), it is only expected to work for the satellite relative motion with small inter-satellite distance.

Further approximation can be made to the linear model of Eq. (2.59) by dropping the second-order J_2 terms. By removing ω_x^2 and replacing steering acceleration α_x with

$$\alpha_x \approx -\frac{k_{J2}s_{2i}c_\theta}{r^5} + \frac{3\dot{r}k_{J2}s_{2i}s_\theta}{r^4 h} \quad (2.63)$$

and substituting $(n^2, \omega_x, \omega_z, \alpha_z)$ in Eqs. (2.43), (2.19), (2.25), and (2.27) into Eq. (2.59), the first-order J_2 linear model is derived as

$$\begin{aligned} \ddot{x}_j &= \frac{2\dot{y}_j h}{r^2} + x_j \left(\frac{2\mu}{r^3} + \frac{h^2}{r^4} + \frac{4k_{J2}(1 - 3s_i^2 s_\theta^2)}{r^5} \right) \\ &\quad - y_j \left(\frac{2\dot{r}h}{r^3} - \frac{3k_{J2}s_i^2 s_{2\theta}}{r^5} \right) + \frac{5k_{J2}z_j s_{2i} s_\theta}{r^5} + a_{jx} \\ \ddot{y}_j &= -\frac{2\dot{x}_j h}{r^2} - \frac{2k_{J2}\dot{z}_j s_{2i} s_\theta}{r^3 h} + x_j \left(\frac{2\dot{r}h}{r^3} + \frac{5k_{J2}s_i^2 s_{2\theta}}{r^5} \right) \\ &\quad - y_j \left(\frac{\mu}{r^3} - \frac{h^2}{r^4} + \frac{k_{J2}(1 + 2s_i^2 - 7s_i^2 s_\theta^2)}{r^5} \right) + z_j \left(\frac{3k_{J2}\dot{r} s_{2i} s_\theta}{r^4 h} - \frac{2k_{J2}s_{2i} c_\theta}{r^5} \right) + a_{jy} \\ \ddot{z}_j &= \frac{2k_{J2}\dot{y}_j s_{2i} s_\theta}{r^3 h} + \frac{5k_{J2}x_j s_{2i} s_\theta}{r^5} - \frac{3k_{J2}y_j \dot{r} s_{2i} s_\theta}{r^4 h} \\ &\quad - z_j \left(\frac{\mu}{r^3} + \frac{k_{J2}(3 - 2s_i^2 - 5s_i^2 s_\theta^2)}{r^5} \right) + a_{jz} \end{aligned} \quad (2.64)$$

The above equation can be rewritten into the following form

$$\frac{d}{dt} \begin{bmatrix} \dot{x}_j \\ \dot{y}_j \\ \dot{z}_j \end{bmatrix} = A_3(t) \begin{bmatrix} \dot{x}_j \\ \dot{y}_j \\ \dot{z}_j \end{bmatrix} + A_4(t) \begin{bmatrix} x_j \\ y_j \\ z_j \end{bmatrix} + \begin{bmatrix} a_{jx} \\ a_{jy} \\ a_{jz} \end{bmatrix} \quad (2.65)$$

where

$$A_3(t) = \begin{bmatrix} 0 & \frac{2h}{r^2} & 0 \\ -\frac{2h}{r^2} & 0 & -\frac{2k_{J2}s_{2i}s_{2\theta}}{r^3h} \\ 0 & \frac{2k_{J2}s_{2i}s_{2\theta}}{r^3h} & 0 \end{bmatrix} \quad (2.66)$$

$$A_4(t) = \begin{bmatrix} \frac{2\mu}{r^3} + \frac{h^2}{r^4} + \frac{4k_{J2}(1-3s_i^2s_\theta^2)}{r^5} & -\frac{2\dot{r}h}{r^3} + \frac{3k_{J2}s_i^2s_{2\theta}}{r^5} & \frac{5k_{J2}s_{2i}s_{2\theta}}{r^5} \\ \frac{2\dot{r}h}{r^3} + \frac{5k_{J2}s_i^2s_{2\theta}}{r^5} & -\frac{\mu}{r^3} + \frac{h^2}{r^4} - \frac{k_{J2}(1+2s_i^2-7s_i^2s_\theta^2)}{r^5} & \frac{3k_{J2}\dot{r}s_{2i}s_{2\theta}}{r^4h} - \frac{2k_{J2}s_{2i}c_\theta}{r^5} \\ \frac{5k_{J2}s_{2i}s_{2\theta}}{r^5} & -\frac{3k_{J2}\dot{r}s_{2i}s_{2\theta}}{r^4h} & -\frac{\mu}{r^3} - \frac{k_{J2}(3-2s_i^2-5s_i^2s_\theta^2)}{r^5} \end{bmatrix} \quad (2.67)$$

The above result can be presented in the following corollary.

Corollary 2.2 Consider a 2-satellite system of the reference satellite S_0 and the member satellite S_j , in the presence of spherical gravity and J_2 gravity of the Earth, relative motion model in the LVLH coordinate can be further simplified into linear equations as in (2.65)–(2.67) by removal of nonlinear effect and second order J_2 terms.

Remark 2.4 The relative motion dynamics described by (2.60) and (2.65) are both linear time-varying system. The time-varying parameters in $A_1(t)$ and $A_2(t)$, $A_3(t)$ and $A_4(t)$ are the parameters of the reference satellite and are explicitly expressed in terms of RSV $(r, \dot{r}, h, i, \theta)$.

Remark 2.5 Both the nonlinear effect and the second-order J_2 effect have been removed in dynamics of Eq. (2.64). As a result, it is simpler than the exact nonlinear model in Eq. (2.47) and the complete linear model in Eq. (2.59). Moreover, in the dynamics of Eq. (2.64), all time-varying parameters of the reference orbit are explicitly expressed in terms of RSV $(r, \dot{r}, h, i, \theta)$.

2.4 Validation of Proposed Dynamic Models by Simulation

Simulations are carried out in MATLAB for three newly developed models, i.e., the exact J_2 nonlinear model in Eq. (2.47), the complete J_2 linear model in Eq. (2.59), and the first-order J_2 linear model in Eq. (2.64). In the simulations, each newly

developed model is compared with an exact J_2 propagator of satellite relative motion, which actually works by taking the difference of integrated solutions of two absolute J_2 dynamics in ECI frame. To do the comparison, the same initial conditions of the reference and member satellites are applied to the exact J_2 propagator and the newly developed models that are combined with RSV dynamics of Eqs. (2.20), (2.21), (2.23), and (2.24).

Different low Earth reference orbits are simulated by changing the initial values of osculating orbital elements on eccentricity and inclination, while the initial values of other osculating orbital elements remain the same as

$$a(0) = 7100 \text{ km}, \omega(0) = -20^\circ, f(0) = 20^\circ, \Omega(0) = 0 \quad (2.68)$$

It is known that the projected circular orbit (PCO) of satellite relative motion does not exist around an eccentric and perturbed reference orbit. However, when the eccentricity is small, the satellite relative orbit may nearly be a circle. Those relative orbits are referred as the Quasi-PCO. In the model simulations, Quasi-PCO is used as the motion of the member satellite, whose initial conditions are given by

$$x_j(0) = \rho/2, \quad y_j(0) = 0, \quad x_j(0) = \rho, \quad \dot{x}_j(0) = 0, \quad \dot{z}_j(0) = 0 \quad (2.69)$$

and

$$\dot{y}_j(0) \text{ is the solution of the energy matching condition.} \quad (2.70)$$

In Eq. (2.69), ρ is the initial radius of Quasi-PCO. Under conditions of Eqs. (2.68) and (2.69), as well as $e = 0.05, i = 45^\circ$ and $\rho = 2.5 \text{ km}$, the Quasi-PCO under J_2 perturbation is shown in Fig. 2.3.

Figures 2.4, 2.5, 2.6, 2.7, 2.8, 2.9, and 2.10 show the simulation results in terms of errors between each newly developed model and the exact J_2 propagator in LVLH frame. Errors of the exact J_2 nonlinear model, the complete J_2 linear model, and the first-order J_2 linear model are presented by dash-dotted lines, dashed lines, and solid lines, respectively.

It can be seen from each figure in Figs. 2.4, 2.5, 2.6, 2.7, 2.8, 2.9, and 2.10 that the model errors of the exact J_2 nonlinear model of Eq. (2.47) are always nearly zero in every scenario. This is because no approximation is taken in the derivation of the exact J_2 nonlinear model, so that it covers all effects of eccentricity, non-linearity, and J_2 perturbation. The simulation results demonstrate that the exact J_2 nonlinear model of Eq. (2.47) is correct and performs perfectly.

Remark 2.6 It is noticed that the primary errors of the two linear models are the drifts in the along-track direction (y). Compared with errors in other two directions (x and z directions, as shown in (a) and (c) of Figs. 2.4, 2.5, 2.6, 2.7, 2.8, 2.9, and 2.10), the error magnitude in the along-track direction (as shown in Fig. 2.10b) is at least one order bigger.

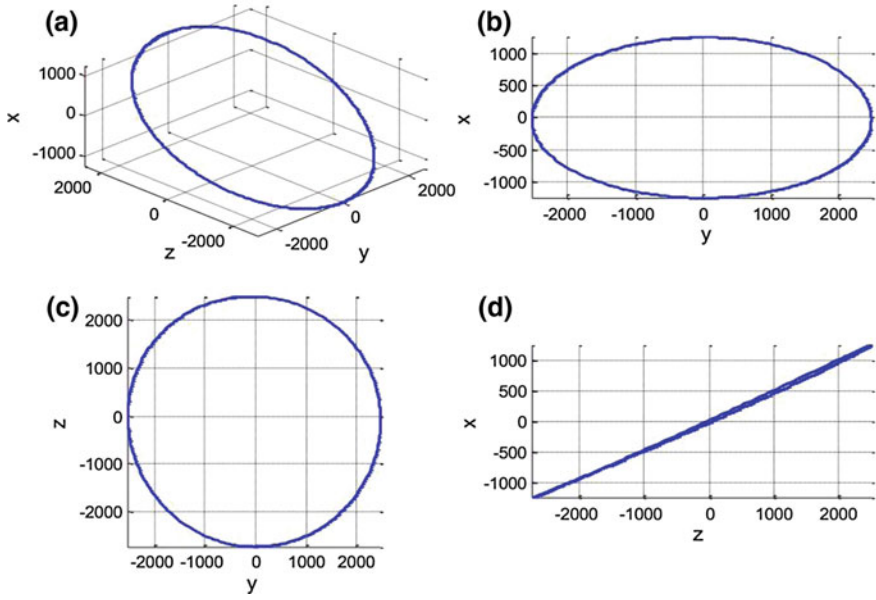
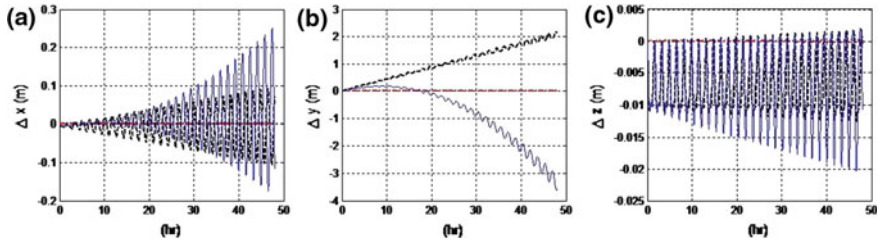
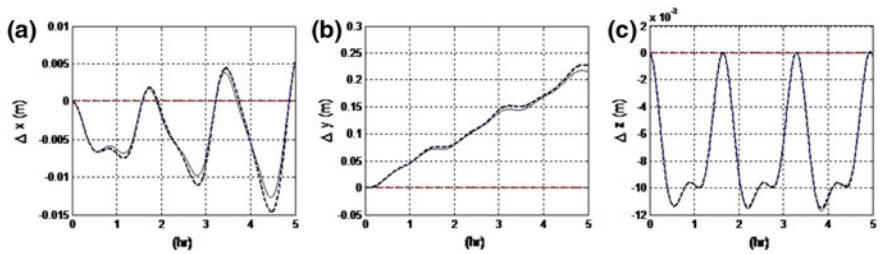


Fig. 2.3 Quasi-PCO of 5 km diameter

Fig. 2.4 Model errors at $\rho = 250$ m, $i = 45^\circ$, $e = 0.05$, $t = 48$ hFig. 2.5 Model errors at $\rho = 250$ m, $i = 15^\circ$, $e = 0.05$, $t = 5$ h

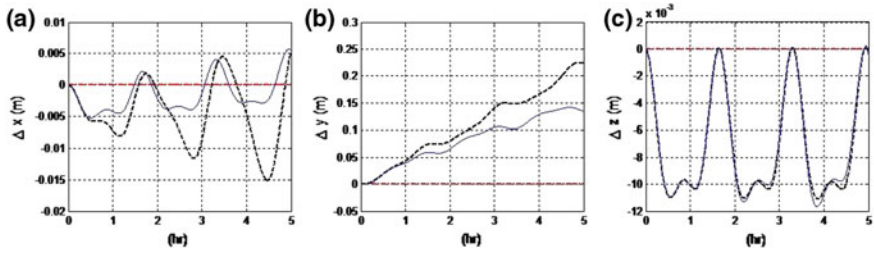


Fig. 2.6 Model errors at $\rho = 250$ m, $i = 45^\circ$, $e = 0.05$, $t = 5$ h

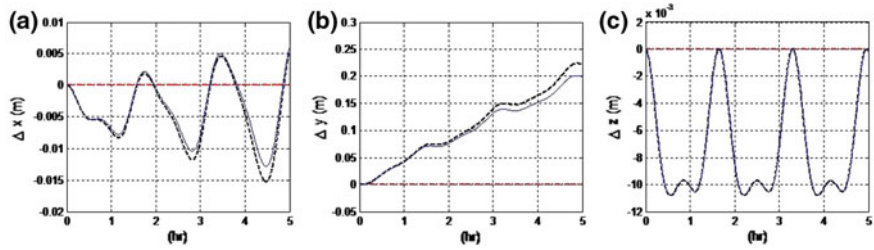


Fig. 2.7 Model errors at $\rho = 250$ m, $i = 85^\circ$, $e = 0.05$, $t = 5$ h

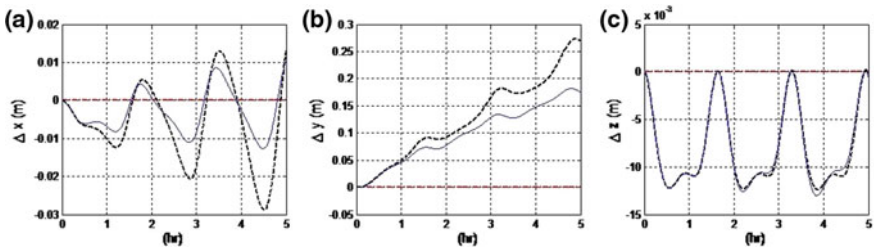


Fig. 2.8 Model errors at $\rho = 250$ m, $i = 45^\circ$, $e = 0.1$, $t = 5$ h

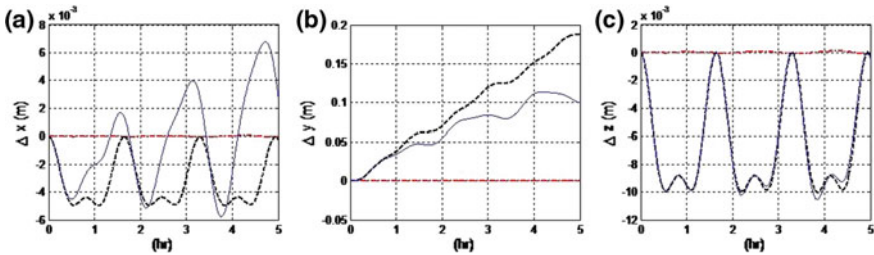


Fig. 2.9 Model errors at $\rho = 250$ m, $i = 45^\circ$, $e = 0$, $t = 5$ h

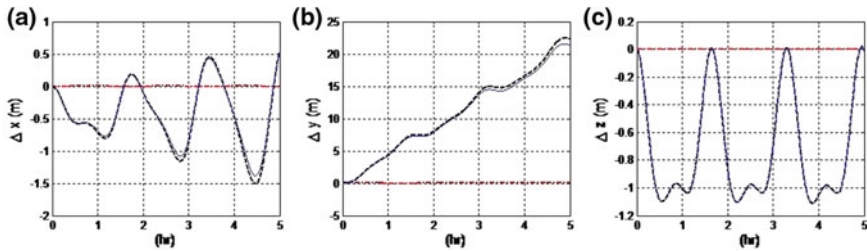


Fig. 2.10 Model errors at $\rho = 2500$ m, $i = 45^\circ$, $e = 0.05$, $t = 5$ h

Next, the drift errors in different satellite flying durations, different orbital inclinations, and eccentricities, as well as different formation sizes, are analyzed.

For a longer satellite flying such as 48 h as shown in Fig. 2.4b, the drift error of the complete J_2 linear model in Eq. (2.59) increases monotonously in positive direction over time, while the drift error of the first-order J_2 linear model in Eq. (2.64) increases firstly in positive direction and then turns down to negative direction. This phenomenon demonstrates that the removed nonlinear effect causes the drift error to increase positively, while the removed second-order J_2 effect is negative. Nevertheless, the secular nonlinear effect is approximately linear over time, while the secular second-order J_2 effect is higher order than the nonlinear effect. Consequently, the nonlinear effect is primary at the beginning so that the drift error increases positively. However, after some hours of flight, the second-order J_2 effect overwhelms the nonlinear effect and becomes dominant. Then, the drift error tends to negative direction.

Actually, the behaviors of the two linear models in the first several hours are more concerned, because the potential application of linear models is the formation maneuvering, which is usually carried out in a relative short duration such as one or few revolution(s). For this reason, analysis are subsequently focused on different scenarios in the first few hours of flight.

Figures 2.5, 2.6, and 2.7 compare the model performances in different inclinations. It is found that the drift errors of two linear models are similar in a smaller inclination (15° as shown Fig. 2.5b) or a bigger inclination (85° as shown Fig. 2.7b), while the drift error of the first-order J_2 linear model in a middle inclination (45° as shown Fig. 2.6b) is smaller. It shows that the second-order J_2 effect becomes dominant later in a smaller or larger inclination orbit and earlier in a middle inclination orbit.

Figures 2.6, 2.7, 2.8, and 2.9 compare the model performances in different eccentricities. It is found that the difference of drift errors between two linear models is smaller in a bigger eccentricity. It demonstrates that the second-order J_2 effect becomes significant earlier in a smaller inclination.

Figures 2.6, 2.7, 2.8, 2.9, and 2.10 compare the model performances in different formation sizes. They show that the two linear models tend to perform worse for a larger satellite formation size. This result agrees with the assumption used to derive the linear models. It is also observed that the second-order J_2 effect becomes evident later for a larger formation.

Remark 2.7 It is concluded that, when satellites fly in low Earth orbits such as a semimajor axis of 7100 km, the removed second-order J_2 effect compensates some removed nonlinear effects in the first several hours of satellite relative motion. So, the first-order J_2 linear model of Eq. (2.64) is more accurate and performs better than the complete J_2 linear model of Eq. (2.59) in a short duration. In particular, the second-order J_2 effect may manifest early and counteract the nonlinear effect more if the satellite flies in a middle inclination orbit which is near circular, and the formation size is small.

2.5 Comparison Study of Relative Dynamic Models

From the literature review in Sect. 2.1, it is known that the complexity of the relative dynamic models increases as the required accuracy becomes greater. The available models were derived under different assumptions and by different methodologies. A comparison study is necessary to select an appropriate model for a specific mission and determine what kind of perturbation should be considered for specific applications. Thus, in this section, a simulation method with a modeling error index is introduced for comparing and evaluating various theories of relative motion of satellite formation dynamics (Hang et al. 2008). The index captures error components of relative position and velocity between a selected satellite relative model and a precise propagator. It is applicable to evaluate the accuracy performance of any valid relative dynamic model including both linear and nonlinear models.

Commercial satellite software Satellite Tool Kit (STK) (www.STK.com) is used as a benchmarking tool to compare various dynamic models. The simulation results show that, when the Earth aspherical gravity and the air drag are present, the accuracy of some models is affected tremendously by eccentricity, semimajor axis, inclination, and formation size. The simulation results may serve for the future dynamic model development and the satellite formation mission design. In this section, the comparison focuses on the direct ODE models (ordinary differential equations in LVLH coordinate). However, the proposed evaluation method is generic and can be applied to other formulations.

2.5.1 Comparison Method with Model Error Index

In this study, all selected dynamic models are compared with a propagator, which is precise, standard, and acceptable in the practical application. In this research, STK is chosen as the key propagator. STK is popular commercial satellite software. It uses advanced algorithm and field data to generate satellite orbit, in which different perturbations can be customized and included.

Figure 2.11 shows the function chart of the model comparison method. In the first step, a group of initial conditions are assigned for both chief and deputy satellites,

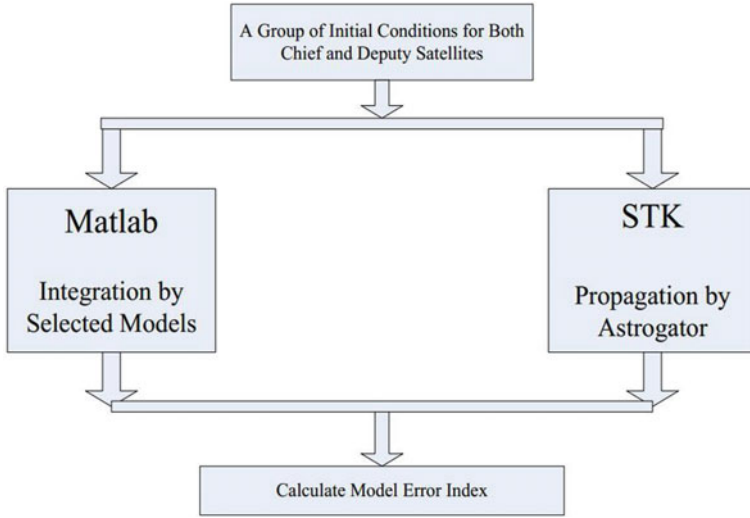


Fig. 2.11 Function chart of model comparison method

including relative positions and relative velocities. In the second step, the STK–MATLAB interface is utilized to transfer data into STK/Astrogator. Then, the initial conditions are propagated in STK which includes various orbital disturbances. Meanwhile, the same initial conditions are integrated numerically by the selected relative dynamics in MATLAB. In the third step, through the STK–MATLAB interface again, the propagated satellite orbit data in STK are collected in MATLAB. Finally, the propagated data from STK and the integrated data by the selected dynamics in MATLAB are synthesized, and the model error index is calculated.

In this comparison method, the key technique is the calculation of the model error index, which is comparable between different dynamic models. Alfriend and Yan (2005) proposed a nonlinear index for comparing the accuracy of various dynamic models. Their method is extended from the linear index of Junkins et al. (1996). Here, a simple index is proposed for the comparison by means of the numerical simulation. The proposed index σ is described as follows:

$$\sigma(n) = \frac{1}{n} \sum_{i=1}^n \log_2(1 + P_i^{(e)})(1 + V_i^{(e)})^w \quad (2.71)$$

where

$$V_i^{(e)} = \cos^{-1} \left(\frac{\mathbf{v}_i^{STK} \cdot \mathbf{v}_i^M}{\|\mathbf{v}_i^{STK}\| \|\mathbf{v}_i^M\|} \right) \quad (2.72)$$

$$P_i^{(e)} = \frac{\|\mathbf{r}_i^{STK} - \mathbf{r}_i^M\|}{\rho} \quad (2.73)$$

$\mathbf{v}_i^{STK}$, $\mathbf{v}_i^M$ are velocity vectors and $\mathbf{r}_i^{STK}$, $\mathbf{r}_i^M$ are position vectors. The data propagated in STK and the data integrated in MATLAB are with superscript STK and M, respectively. n denotes the total steps. i denotes the i -th step. $P_i^{(e)}$ and $V_i^{(e)}$ represent the position and velocity differences between the results of STK and dynamic model. In detail, $V_i^{(e)}$ is an adjusted radian value between two velocity vectors. It can effectively indicate the model error caused by the relative orbit rotation such as the tumbling. The exponent w is a weight to adjust $V_i^{(e)}$ in line with $P_i^{(e)}$. Increasing w would lead to magnifying rotation effect in the model error index and vice versa. In the simulation, $w = 2$ is carried out to balance the errors caused by the relative orbit rotation and the drift.

Note that the above model error index is computed at every sampling step. Moreover, this model error index is proportional to the model error. So, a smaller model error index implies that the model is more accurate.

2.5.2 Selected Dynamic Models for Comparison Study

This comparison study focuses on the direct ODE models. Thus, the equations, which describe the relative motion between satellites, are generally in the following form:

$$f(\ddot{\mathbf{q}}, \dot{\mathbf{q}}, \mathbf{q}, \mathbf{e}) = 0 \quad (2.74)$$

where $\mathbf{q} = [x \ y \ z]^T$ represents the vector of relative position of the deputy satellite in LVLH coordinate frame. $\mathbf{e}$ is the vector of orbit parameters of the chief satellite in ECI frame. It is constant for time-invariant models and is variable for time-varying models.

Five widely used models, including Clohessy-Wiltshire (CW) model, Tschauner-Hempel (TH) model, Unperturbed Nonlinear (UN) model, Schweighart-Sedwick (SS) model, and Xu-Wang model, are compared and evaluated. The assumptions about reference (chief) satellite orbit, perturbation, and formation size are different for each model and are summarized in Table 2.1. The five models are introduced as follows:

2.5.2.1 Clohessy-Wiltshire Model

CW model (Hill 1878) (Hill's equations) is established in the LVLH coordinate by making the assumption of a circular chief orbit, spherical Earth, linearizing the

Table 2.1 Comparison of assumptions for direct ODE models

Models		CW	TH	SS	NU	XW
Assumption	Chief orbit	C	E	C	E	E
	Perturbation	No	No	J_2	No	J_2
	Formation size	S	S	S	L	L

Note S and L represent small and large, respectively. C and E denote circle and ellipse, respectively

differential gravitational forces, and no other perturbations included. It was used initially to solve the satellite rendezvous problem and later to study the satellite formation flying. CW model is a fundamental model. Other direct ODE models are benchmarked on this model.

$$\ddot{x} - 2n\dot{y} - 3n^2x = 0 \quad (2.75)$$

$$\ddot{y} + 2n\dot{x} = 0 \quad (2.76)$$

$$\ddot{z} + n^2z = 0 \quad (2.77)$$

where x, y and z are the LVLH Cartesian coordinates, and $\dot{x}, \dot{y}$ and $\dot{z}$ are the relative velocity components, $n = \sqrt{\mu/r^3}$ is the mean motion, μ denotes gravitational coefficient, and r denotes radius of satellite orbit.

2.5.2.2 Tschauner-Hempel Model

Inalhan and How (Tillerson and How 2002; Clohessy and Wiltshire 1960) proved that the eccentricity has great effects on the relative motion. Using CW model, even a small eccentricity of $e = 0.005$ may result in a large fuel consumption to maintain a specified formation. Tschauner and Hempel (1965) presented a method to express the linearized relative dynamics around an eccentric orbit. TH model can be taken as an extension of CW model.

$$\ddot{x} = \frac{2\mu}{r^3}x + 2\omega y + \omega^2x \quad (2.78)$$

$$\ddot{y} = \frac{-\mu}{r^3}y - 2\omega\dot{x} - \omega^2y \quad (2.79)$$

$$\ddot{z} = \frac{-\mu z}{r^3} \quad (2.80)$$

where $\omega = \dot{f} = \frac{n(1+e\cos f)^2}{(1-e^2)^{3/2}}$, $\dot{\omega} = \ddot{f} = \frac{-2n^2e\sin f(1+e\cos f)^3}{(1-e^2)^3}$, $r = \frac{a(1-e^2)}{1+e\cos f}$.

Since this linear model will be used in Sect. 5.3 for decentralized formation keeping, this model needs to include the control input and be rewritten into a new form. Although (2.78)–(2.80) is expressed in the time domain, monotonically increasing true anomaly f of the chief orbit provides a natural basis for parameterizing the fleet time and motion. This observation is based on the fact that the angular velocity and the radius describing the orbital motion are functions of the true anomaly. When f is used as the free variable, the equations of motion can be transformed using the following relationships between derivation over time (represented by $(\dot{\cdot})$) and derivation over θ (represented by $(\cdot)'$)

$$(\dot{\cdot}) = (\cdot)' \dot{f}, \quad (\ddot{\cdot}) = (\cdot)'' \dot{f}^2 + \dot{f} \dot{f}' (\cdot)' \quad (2.81)$$

where the time rate of change of the true anomaly $\dot{f}$ can be written as

$$\dot{f} = \frac{n(1 + e \cos f)^2}{(1 + e^2)^{3/2}} \quad (2.82)$$

With these transformations, the set of LTV equations describing the relative motion of the satellite with respect to an elliptic reference orbit can be written as (Inalhan et al. 2002).

$$\mathbf{x}'(f) = A(f)\mathbf{x}(f) + B(f)\mathbf{u}(f) \quad (2.83)$$

where

$$A(f) = \begin{bmatrix} \frac{2e \sin f}{1 + e \cos f} & \frac{3 + e \cos f}{1 + e \cos f} & 2 & \frac{-2e \sin f}{1 + e \cos f} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & \frac{2e \sin f}{1 + e \cos f} & \frac{2e \sin f}{1 + e \cos f} & \frac{e \cos f}{1 + e \cos f} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2e \sin f}{1 + e \cos f} & \frac{-1}{1 + e \cos f} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad (2.84)$$

$$B(f) = \frac{(1 - e^2)^3}{(1 + e \cos f)^4 n^2} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}^T \quad (2.85)$$

where $\mathbf{x}(f) = [x', x, y', y, z', z]^T$ represent relative positions and velocities of the satellite with respect to the LVLH frame expressed in f domain, $\mathbf{u}(f) = [u_x \ u_y \ u_z]^T$ represent the vector of control acceleration, n is the natural frequency of the reference orbit, and e is the eccentricity.

2.5.2.3 Unperturbed Nonlinear Model

TH model can be further expanded to a system of nonlinear differential equations accounting for both effects of nonlinearity and eccentricity (Gurfil 2005), which is called the unperturbed nonlinear model. The parameter $\mathbf{e}$ of the reference satellite in (2.47) is described by three augmented first-order differential equations in the unperturbed nonlinear model.

$$\ddot{x} - 2\dot{\theta}\dot{y} - \ddot{\theta}y - \dot{\theta}^2x = -\frac{\mu(r_c + x)}{\left[(r_c + x)^2 + y^2 + z^2\right]^{3/2}} + \frac{\mu}{r_c^2} \quad (2.86)$$

$$\ddot{y} + 2\dot{\theta}\dot{x} + \ddot{\theta}x - \dot{\theta}^2y = -\frac{\mu y}{\left[(r_c + x)^2 + y^2 + z^2\right]^{3/2}} \quad (2.87)$$

$$\ddot{z} = -\frac{\mu z}{\left[(r_c + x)^2 + y^2 + z^2\right]^{3/2}} \quad (2.88)$$

$$\ddot{r}_c = r_c \dot{\theta}^2 - \frac{\mu}{r_c^2} \quad (2.89)$$

$$\ddot{\theta} = -\frac{2\dot{r}_c \dot{\theta}}{r_c} \quad (2.90)$$

where x, y and z are the relative motion coordinates of the deputy with respect to the chief in the LVLH frame. r_c refers to the radius of the chief satellite from the center of the Earth, θ refers to the latitude angle of the chief, and μ is the gravitational parameter.

2.5.2.4 Schweighart-Sedwick Model

Schweighart and Sedwick (2002) developed a set of linearized differential equations to capture the relative motion between satellites under J_2 effect around a circular orbit. This model can be taken as another extension of CW model.

$$\ddot{x} - 2(nc)\dot{y} - (5c^2 - 2)n^2x = 0 \quad (2.91)$$

$$\ddot{y} + 2(nc)\dot{x} = 0 \quad (2.92)$$

$$\dot{\gamma} - nb \cos \gamma = 0 \quad (2.93)$$

$$\dot{\Phi} - nb\Phi \cos \gamma \sin \gamma = 0 \quad (2.94)$$

where $z = r_{\text{ref}}\Phi \sin(knt - \gamma)$. γ is angular distance between the equator and the intersection of two orbital planes (chief and deputy orbit). Φ represents maximum angular cross-track separation. $r_{\text{ref}}, i_{\text{ref}}$ are radius and inclination of reference satellite orbit.

2.5.2.5 Xu-Wang Model

Xu and Wang (2008a) recently developed a satellite relative dynamic model which includes eccentricity, nonlinearity, and J_2 perturbation. This model is developed based on the essential fact that the precise relative dynamics highly depends on the accurate information of the reference orbit. The parameter $\mathbf{e}$ of the chief satellite in (2.74) is described by five augmented first-order differential equations to describe reference orbit dynamics. It is stated that this dynamic model does not have model error in arbitrary eccentric orbits under J_2 perturbation. So, Xu-Wang model can be used to propagate the satellite relative motion from arbitrary initial conditions. Xu-Wang model is introduced in Sect. 2.2.2.

2.5.3 Case Studies

2.5.3.1 Simulation Scenario

For setting up comparison scenario, the idea of the projected circular orbit (PCO) in the LVLH frame centered at the chief satellite is utilized. It can be described by

$$x = 0.5\rho \sin(\theta + \alpha_0) \quad (2.95)$$

$$y = \rho \cos(\theta + \alpha_0) \quad (2.96)$$

$$z = \rho \sin(\theta + \alpha_0) \quad (2.97)$$

where ρ is the radial of PCO; θ is the true latitude angle of the chief satellite; α_0 is the initial phase angle of relative orbit. It is known that an exact PCO is only possible in circular orbits. When eccentricity and perturbations are present, the relative orbit may be distort. However, if the eccentricity is small, the relative orbit will still be close to PCO. Actually, an exact PCO is not necessary in the comparison study, and a distorted PCO is acceptable.

In all test cases, the energy-matching initial conditions derived by Xu and Wang (Xu et al. 2009) are used to prevent a fast drift of the formation. It should be kept in mind that both the distorted PCO formation and the energy-matching conditions do not change the evaluation and comparison results. In the simulation, no matter what

Table 2.3 Orbit parameters of deputy satellite

Deputy satellite orbit	Value
x_0	0.5 km
y_0	0 km
z_0	ρ km
$\dot{x}_0$	0 km/s
$\dot{y}_0$	Adjusted
$\dot{z}_0$	0 km/s
ρ	0.10–20.0 km

Table 2.2 Orbit elements of chief satellite

Chief satellite orbit	Value
a	6600–8000 km
Ω	0°
i	0° – 90°
e	0–0.01
ω	0°
v	0°

Table 2.4 Physical parameters of both satellites

Satellite parameters	Value
Dry mass	150 kg
Drag coefficient	2.2
Drag area	5 m^2

kind of formation is used, the selected dynamic model should predict the relative motion which is close to the result in the STK propagator as long as the same set of initial conditions are applied. Nevertheless, using a stable formation may improve the reliability of the model error index.

Four parameters i , e , a , and ρ are selected as variables for comparison in the simulation. Later on, it will be proved that these four parameters have important effects on dynamic model accuracy. The objective was to calculate the model error index σ of each selected model due to each of the variables e , a , i , and ρ . In the simulation, the testing cases are assigned in the following Tables 2.2, 2.3, and 2.4:

Within the calculation of the model error index by (2.71), the scenario duration is 24 h, and the sampling interval is 1 min, which correspond that the total steps are $n = 24 \times 60 = 1440$. The astrogator in STK is customized to incorporate the Earth zonal harmonic perturbations up to J_{21} and the atmospheric drag.

2.5.3.2 Case 1: Error Index Versus Formation Size

Figure 2.12 shows the index comparison varying with the PCO radius. It is clear that the model error indexes of linear models tend to be larger than those of nonlinear models (unperturbed nonlinear model and Xu-Wang model) as the

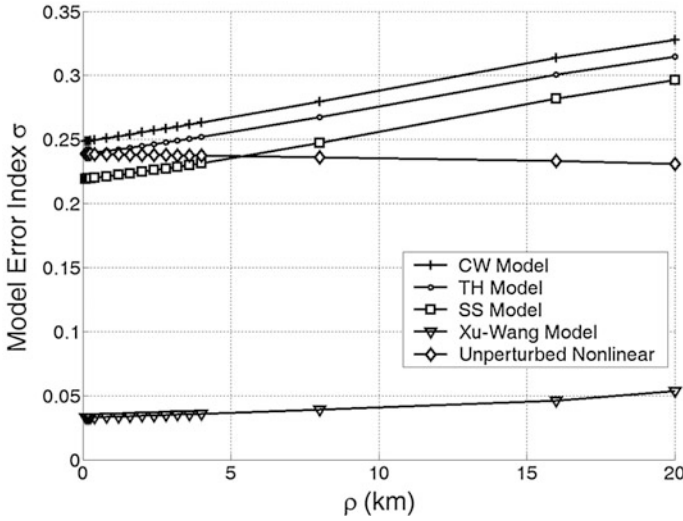


Fig. 2.12 Index comparison for $e = 0.0001$, $i = 45^\circ$, $a = 6600$ km

formation size increases. In this simulation, since the eccentricity is almost zero, the indexes of CW model and TH model are close to each other, while SS model that include J_2 effects performs better than other linear models. The model error difference between TH model and unperturbed nonlinear model in the figure shows the impact of nonlinearity. It is a function of formation size and increases as formation size getting large. The model error difference between unperturbed nonlinear model and Xu-Wang model shows the impact of J_2 effects. It is found that the influence of J_2 effects is bigger than the nonlinearity effect.

2.5.3.3 Case 2: Error Index Versus Eccentricity

Figure 2.13 shows the error index as a function of the eccentricity. It can be seen that errors of the models which exclude eccentricity consideration (CW model and SS model) grow larger and larger as the eccentricity gets larger. Since the formation size is small, the performance of TH model is almost the same as unperturbed nonlinear model, because both of them compensate the effect of eccentricity. In contrast to error due to formation size variation, eccentricity changes lead to more noticeable error growth. The model error difference between CW model and TH model shows the impact of eccentricity. In contrast to the error difference between the unperturbed nonlinear model and Xu-Wang model, which shows impact of J_2 effects, it is found that the eccentricity leads to much larger error growth. Thus, eccentricity is the dominant error in formation flying design and needs to be considered with high priority.

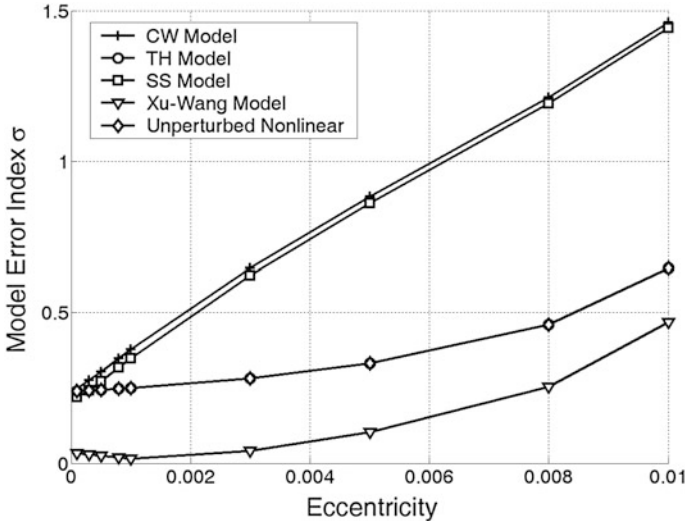


Fig. 2.13 Index comparison for $\rho = 0.1$ km, $i = 45^\circ$, $a = 6600$ km

2.5.3.4 Case 3: Error Index Versus Inclination

Figure 2.14 shows error index comparison as a function of the inclination. The simulation results show that SS model has distinct error trend. It is interesting that the SS model even does not perform as good as the CW model in a larger inclination. The error indexes of all other models decrease as inclination increases. Nevertheless, the models, which consider J_2 effects, have better performance than other models (the change in magnitude is relatively small.). This demonstrates that the J_2 effect has a tight relationship with the inclination.

2.5.3.5 Case 4: Error Index Versus Semimajor Axis

Figure 2.15 shows error index comparison as a function of the chief satellite perigee. Since none of the selected dynamic models considers the atmospheric drag, their model errors increase exponentially when the chief satellite perigee is smaller than a certain value. It can be seen that the inflexion of model error indexes appears when the perigee is around 300 km, which corresponds to a satellite semimajor axis of about 6750 km. Therefore, if the satellite altitude is lower than 300 km, the inclusion of atmospheric drag in the model is necessary. On the other hand, if the satellite altitude is higher than 400 km, the effect of atmospheric drag is small.

In summary, the index comparison shows every model has distinct characteristic and performance, which is summarized in Table 2.5.

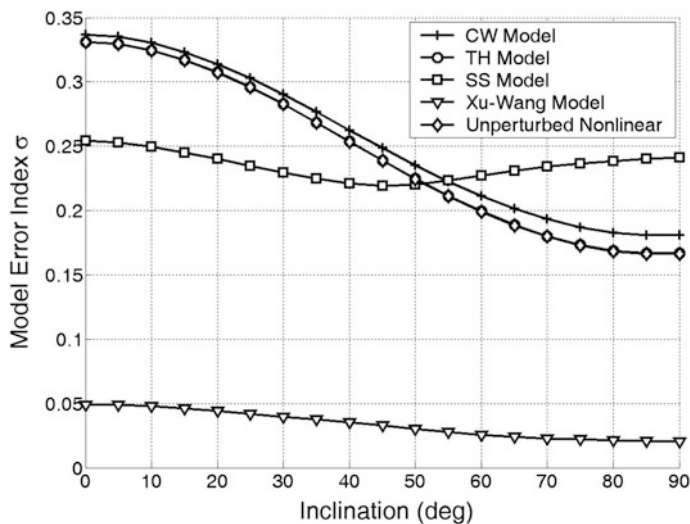


Fig. 2.14 Index comparison for $\rho = 0.1$ km, $e = 0.0001$, $a = 6600$ km

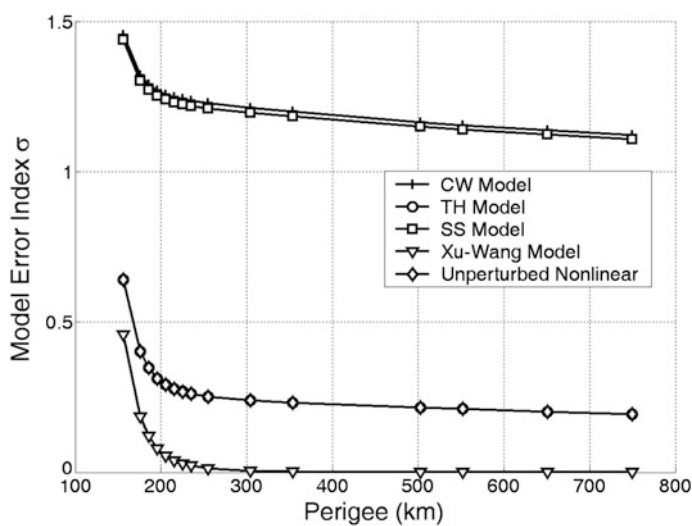


Fig. 2.15 Index comparison for $\rho = 0.1$ km, $e = 0.01$, $i = 45^\circ$

Table 2.5 Comparison of model error sources for direct ODE models

Models		CW	TH	SS	NU	XW
Model error caused by	Eccentricity	L	S	L	S	S
	J_2	L	L	L	L	S
	Nonlinearity	L	L	L	S	S

Note “S” and “L” represent small and large, respectively

2.6 Summary

Firstly, the exact J_2 dynamics for a single satellite is presented based on RSV in this chapter. The expression of this dynamics itself is rather simple. Three variables of RSV, i.e., angular momentum h , inclination i , and right ascension of ascending node Ω , are slowly time varying under perturbations. The most important benefit of this dynamics is that the rotation properties of LVLH frame can be expressed in terms of RSV explicitly and simply. So, it is suitable to describe the motion of a rotating LVLH frame.

Secondly, the exact J_2 nonlinear relative model is derived. Since no approximation is applied in the derivation, this model does not have error if only J_2 perturbation is present. The model depends on 5 parameters of RSV, so that the satellite relative motion is equivalently described by 11 simple first-order differential equations. It is a good candidate to solve the precise control problems of long-term flying satellite formation such as formation maintenance. It can also be used in the study of formation design or even as a propagator of satellite relative motion.

Thirdly, two J_2 linear relative models are developed by removing the nonlinear effect and the second-order J_2 effect from the exact J_2 nonlinear relative model. The numerical study demonstrates that the removed second-order J_2 effect compensates some of the removed nonlinear effect at the beginning stage. Moreover, the first-order J_2 linear relative model is much simpler than the other two models. It is expected to have potential application on formation maneuver and keeping.

Lastly, a simulation-based evaluation method is presented to compare various satellite formation flying dynamic models in this chapter. With the formulation of a model error index, five existing direct ODE models have been evaluated and compared in the proposed simulation platform. The simulation results provide insights and guidance for the selection of suitable relative dynamic model taking into account the presence of various perturbation effects. The numerical results show that eccentricity, J_2 perturbation, nonlinearity, and atmospheric drag play different roles to affect the model error. The decision on model selection should be made through the balanced consideration of every factor. If the satellite formation flies in orbits higher than 400 km, the eccentricity dominates the disturbances; the next crucial perturbation is J_2 effect followed by nonlinearity. However, if the satellite formation flies in orbits lower than 300 km, the atmospheric drag affects the model accuracy significantly. In contrast to other factors, the nonlinearity has a

smaller effect for a small size formation. Among the five selected direct ODE models, the developed, exact nonlinear relative dynamics Xu-Wang model, introduced in Sect. 2.2.2, which takes into account of both J_2 and chief (reference) orbit eccentricity, performs best in every simulated scenario.

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Chapter 3

Passive and Periodic Satellite Formation Design Around an Oblate Earth

Passive and periodic relative trajectories (PPRT) are a practical option for long-term satellite formation flying. This chapter proves that such trajectories under the J_2 perturbation are only possible under two circumstances. The first circumstance is that all satellites fly in the orbits at the critical inclination. It is revealed that passive and periodic relative orbits with cross-track motion are only possible at the critical inclination under conditions that all satellites share the same inclination, the same semimajor axis, and the same eccentricity. Furthermore, a quasi-periodic relative orbit is achieved near the critical inclination by matching the orbital energy and the polar component of the orbital angular momentum. The second circumstance is that all satellites fly in the same regressing orbital plane. The satellites are further subject to four constraints: the same inclination, semimajor axis, eccentricity, and ascending node. Investigation of this circumstance focuses on the Almost Constant Separation Formation. It is revealed that under a special geometric constraint on the remaining two differential orbit elements, the relative distance between satellites may change very little even in highly elliptic orbits. Results in this chapter are valid in arbitrary elliptic orbits under the J_2 perturbation.

3.1 Introduction

A passive and periodic satellite formation is critical to sustain a long-term formation flying mission. It is proved in Gurfil (2005), Xing et al. (2007) that the passive and periodic satellite relative trajectories in the classical Keplerian motion around an ideal spherical Earth can be achieved as long as every satellite in the formation has the same orbital period T , which is equivalent to having the same mean motion $\dot{M}$, or same semimajor axis a , or same orbital energy E :

$$\delta T = 0 \Leftrightarrow \delta \dot{M} = 0 \Leftrightarrow \delta a = 0 \Leftrightarrow \delta E = 0 \quad (3.1)$$

Different formations around an ideal spherical Earth were proposed for a circular reference orbit (Sabot et al. 2001) and for an elliptic reference orbit (Lane and Axelrad 2006; Jiang et al. 2008).

However, various perturbations around the Earth require a formation flying satellite expel fuel to overcome the perturbations to remain in formation. Among the perturbations, the J_2 gravity is the dominant factor and consumes a large amount of fuel that renders long-term formation flying unsustainable. A passive and periodic satellite formation accommodating the J_2 perturbation will ensure long-term flying formation becomes practically possible.

Including the J_2 gravity term into the classical Keplerian motion causes the secular growths of three mean orbit elements $(\bar{\Omega}^t, \bar{\omega}^t, \bar{M}^t)$, i.e., the right ascension of the ascending node, the argument of perigee, and the mean anomaly (here, the over bars denote the mean orbit elements, and the superscript t denotes the time-varying variables). It is known that the rates $(\dot{\bar{\Omega}}, \dot{\bar{\omega}}, \dot{\bar{M}})$ of the three secular growths are linear under the J_2 perturbation. Schaub et al. (Schaub and Alfriend 2001; Schaub and Junkins 2003) affirmed that, to keep the satellites from drifting apart, it would be desirable to match the three linear rates for all satellites in a given formation. They also show that matching the three linear rates is equivalent to matching another three constant mean orbit elements $(\bar{a}, \bar{e}, \bar{i})$, i.e., the semimajor axis, the eccentricity, and the inclination:

$$\begin{cases} \delta \dot{\bar{\Omega}} = 0 \\ \delta \dot{\bar{\omega}} = 0 \\ \delta \dot{\bar{M}} = 0 \end{cases} \Leftrightarrow \begin{cases} \delta \bar{a} = 0 \\ \delta \bar{e} = 0 \\ \delta \bar{i} = 0 \end{cases} \quad (3.2)$$

Conditions (3.2) are referred as the strict J_2 invariant trajectory conditions. Since these conditions are very conservative and severely restrict the possible relative trajectories, Schaub et al. (Schaub and Alfriend 2001; Schaub and Junkins 2003) further proposed the so-called J_2 invariant trajectory conditions which impose only two constraints, i.e., matching the rate $\dot{\bar{\Omega}}$ of right ascension of ascending node and the sidereal orbital period (i.e., the rate $\dot{\bar{M}} + \dot{\bar{\omega}}$):

$$\delta \dot{\bar{\Omega}} = 0 \quad \delta \dot{\bar{M}} + \delta \dot{\bar{\omega}} = 0 \quad (3.3)$$

Comparing with the strict J_2 invariant trajectory conditions (3.2), the J_2 invariant trajectory conditions (3.3) release one more degree of freedom for satellite formation geometry design. However, the J_2 invariant trajectory conditions (3.3) are still deemed rather conservative such that a formation with cross-track motion cannot be designed at will. Consequently, some researchers (Tragesser and Skreheart 2008) suggested confining a formation by only one constraint, i.e., matching the nodal orbital period only

$$\delta \dot{M} + \delta \dot{\omega} + \delta \dot{\Omega} \cos \bar{i} = 0 \quad (3.4)$$

This condition is known as the no-drift condition. On one side, this condition can be taken as the general case of the period-matching condition (3.1) of the unperturbed Keplerian motion where the nodal rate $\dot{\Omega}$ and the rate $\dot{\omega}$ of argument of perigee are constant zero. Also, since both the strict J_2 invariant trajectory conditions (3.2) and the J_2 invariant trajectory conditions (3.3) ensure the no-drift condition (3.4), the no-drift condition (3.4) can also be viewed as a general case of another two J_2 invariant trajectory conditions of (3.2) and (3.3).

An observation to the above-mentioned conditions under J_2 perturbation is as follows. Although these conditions may somewhat prevent formation flying satellites from drifting apart, none of them can guarantee a satellite under J_2 perturbation to track a passive and periodic relative trajectory. This observation is important as it implies that a satellite that satisfies the strict J_2 invariant trajectory conditions (3.2) may still tumble in its relative trajectory and requires compensation to maintain the periodic relative trajectory. Coincidentally and independently, this observation has been illustrated recently by numerical study (Sabatini et al. 2008).

In this chapter, the general conditions are established for a passive and periodic relative trajectory under the J_2 perturbation (Xu et al. 2009b). It is shown that there are two classes of PPRT. The first class is that all satellites fly in the critical inclination, and the second class is that all satellites fly in the same regressing orbital plane. Further investigations focus on the formation geometry design based on the second class of passive and periodic formations. A special formation of the second class is the Almost Constant Separation Formation (Xu et al. 2009a). This formation imposes an additional geometric constraint that the ratio of the differential mean anomaly to the differential argument of perigee is a constant value associated with the eccentricity. The only available degree of freedom is used to specify the distance between satellites.

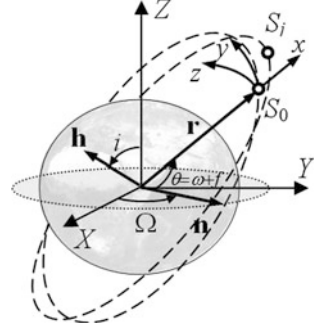
The rest of this chapter is organized as follows. In Sect. 3.2, the general conditions of the passive and periodic relative trajectory are described and proved. In Sect. 3.3, periodic and quasi-periodic relative orbits at critical inclination are investigated. In Sect. 3.4, the formation in the same orbital plane is investigated. In Sect. 3.5, the conclusions are summarized.

3.2 Passive and Periodic Relative Motion Under J_2 Perturbation

In this section, the general conditions of the passive and periodic relative motion under J_2 perturbation are investigated.

In the study of the satellite relative motion, without loss of generality, a 2-satellite system is considered (as shown in Fig. 3.1), i.e., a passive or a virtual reference satellite S_0 (without control force) and a member satellite S_j (with or without control force).

Fig. 3.1 Curvilinear coordinate



The relative motion of the member satellite S_j can be presented in the local curvilinear coordinate xyz in terms of the orbit elements (a, e, i, ω, f) of the reference satellite S_0 and the differential orbit elements $(\delta a_j, \delta e_j, \delta i_j, \delta \omega_j, \delta \Omega_j, \delta M_j)$ between the reference satellite and the j th member satellite. In the literature, the expressions of such satellite relative motion have been derived by Schaub (Schaub and Junkins 2003; Schaub 2004), Lane and Axelrad (2006), as well as Hamel and Lafontaine (2007).

The result of Hamel and Lafontaine (2007) is as follows:

$$\begin{aligned} x_j &= \frac{r(f)}{a} \delta a_j - a c_f \delta e_j + \frac{a e s_f}{\eta} \delta M_j \\ y_j &= \frac{r(f)}{\eta^2} s_f (2 + e c_f) \delta e_j + r(f) c_i \delta \Omega_j + r(f) \delta \omega_j + \frac{r(f)}{\eta^3} (1 + e c_f)^2 \delta M_j \\ z_j &= r(f) s_{f+\omega} \delta i_j - r(f) s_i c_{f+\omega} \delta \Omega_j \end{aligned} \quad (3.5)$$

where $s_{(\cdot)} = \sin(\cdot)$ and $c_{(\cdot)} = \cos(\cdot)$. In (3.5), r is the distance from the reference satellite to the Earth center,

$$r(f) = \frac{a \eta^2}{1 + e c_f} \quad (3.6)$$

with η being another expression of the eccentricity

$$\eta = \sqrt{1 - e^2} \quad (3.7)$$

The Eq. (3.5) contains 11 parameters of which (a, e, i, ω, f) are the osculating orbit elements of the reference satellite S_0 , and $(\delta a_j, \delta e_j, \delta i_j, \delta \omega_j, \delta \Omega_j, \delta M_j)$ are the osculating differential orbit elements of the member satellite S_j . It is noted that the result (3.5) is linear to the differential orbit elements and is independent of the right ascension of ascending node Ω of the reference satellite S_0 . The independence property of the right ascension of ascending node is identical to the exact satellite relative dynamics under J_2 perturbation obtained in Xu and Wang (2008). However, this property is due to somewhat different reasons. Actually, a precise result of

satellite relative motion depends on Ω of the reference satellite S_0 . From theoretical point of view, it should appear in the result when the accelerations that depend on Ω are present (e.g., the sectorial and tesseral harmonic perturbations). Fortunately, these perturbations are small and their impacts on the accuracy of the relative motion are insignificant. The omission of Ω in the linear result (3.5) is caused by the approximation used to evaluate the osculating radius in Eq. (3.6). Coincidentally, this approximation does not affect the inclusion of the zonal harmonic perturbations that are independent of Ω .

Next, the secular relative motion of member satellite S_j under J_2 perturbation is investigated. From the well-known J_2 perturbation theory, there are three types of changes in the osculating orbit elements, namely short-period and long-period oscillations and secular growth. Among three types of changes, the secular growth is significant to the relative motion. Because the J_2 perturbation theory manifests that the secular values of (a, e, i) and $(\delta a_j, \delta e_j, \delta i_j)$ are constants, these elements may be approximated by secular values as

$$a \approx \bar{a} \quad i \approx \bar{i} \quad e \approx \bar{e} \quad (\eta \approx \bar{\eta}) \quad (3.8)$$

and

$$\delta a_j \approx \delta \bar{a}_j \quad \delta i_j \approx \delta \bar{i}_j \quad \delta e_j \approx \delta \bar{e}_j \quad (3.9)$$

where the over bars denote the secular part of a variable. On the other hand, the secular values of (ω, f) and $(\delta \omega_j, \delta \Omega_j, \delta M_j)$ are time-varying and may be approximated as

$$\omega \approx \bar{\omega}^t \quad f \approx \bar{f}^t \quad (3.10)$$

and

$$\delta \omega_j \approx \delta \bar{\omega}_j^t \quad \delta \Omega_j \approx \delta \bar{\Omega}_j^t \quad \delta M_j \approx \delta \bar{M}_j^t \quad (3.11)$$

where the superscript t denotes the time-varying variables. Considering (3.6) and substituting (3.8)–(3.11) into (3.5), the secular relative motion of the member satellite under J_2 perturbation can be expressed as

$$\begin{aligned} x_j(\bar{f}^t, \delta \bar{M}_j^t) &= \frac{\bar{\eta}^2 \delta \bar{a}_j}{1 + \bar{e} c_{\bar{f}^t}} - \bar{a} \delta \bar{e}_j \cdot c_{\bar{f}^t} + \frac{\bar{a} \cdot \bar{e}}{\bar{\eta}} s_{\bar{f}^t} \delta \bar{M}_j^t \\ y_j(\bar{f}^t, \delta \bar{M}_j^t, \delta \bar{\Omega}_j^t, \delta \bar{\omega}_j^t) &= \frac{2 + \bar{e} c_{\bar{f}^t}}{1 + \bar{e} c_{\bar{f}^t}} \bar{a} \delta \bar{e}_j \cdot s_{\bar{f}^t} + \frac{\bar{a}}{\bar{\eta}} (1 + \bar{e} c_{\bar{f}^t}) \delta \bar{M}_j^t + \frac{\bar{a} \bar{\eta}^2}{1 + \bar{e} c_{\bar{f}^t}} (c_i \delta \bar{\Omega}_j^t + \delta \bar{\omega}_j^t) \\ z_j(\bar{f}^t, \delta \bar{\Omega}_j^t, \bar{\omega}^t) &= \frac{\bar{a} \bar{\eta}^2}{1 + \bar{e} c_{\bar{f}^t}} (\delta \bar{i}_j \cdot s_{(\bar{f}^t + \bar{\omega}^t)} - s_{\bar{f}^t} c_{(\bar{f}^t + \bar{\omega}^t)} \delta \bar{\Omega}_j^t) \end{aligned} \quad (3.12)$$

Basically, secular relative motion (3.12) is a very complicated trajectory under the J_2 perturbation. Here, just the strict J_2 invariant relative orbit is investigated. As proved by Schaub and Alfriend (2001), the member satellite and the reference satellite have the same semimajor axis, the same inclination, and the same eccentricity

$$\delta \bar{a}_j = 0 \quad \delta \bar{i}_j = 0 \quad \delta \bar{e}_j = 0 \quad (3.13)$$

and the differences of another three orbit elements

$$\delta \bar{\omega}_j^t = \delta \bar{\omega}_j \quad \delta \bar{\Omega}_j^t = \delta \bar{\Omega}_j \quad \delta \bar{M}_j^t = \delta \bar{M}_j \quad (3.14)$$

are constants. Substituting (3.13) and (3.14) into (3.12) yields the solution of the strict J_2 invariant relative orbit

$$\begin{aligned} x_j(\bar{f}^t) &= \frac{\bar{a} \cdot \bar{e} \delta \bar{M}_j}{\bar{\eta}} \cdot s_{\bar{f}^t} \\ y_j(\bar{f}^t) &= \frac{\bar{a} \delta \bar{M}_j}{\bar{\eta}} (1 + \bar{e} c_{\bar{f}^t}) + \frac{\bar{a} \bar{\eta}^2 (c_{\bar{i}} \delta \bar{\Omega}_j + \delta \bar{\omega}_j)}{1 + \bar{e} c_{\bar{f}^t}} \\ z_j(\bar{f}^t, \bar{\omega}^t) &= -\bar{a} \bar{\eta}^2 s_{\bar{i}} \delta \bar{\Omega}_j \cdot \frac{c_{(\bar{f}^t + \bar{\omega}^t)}}{1 + \bar{e} c_{\bar{f}^t}} \end{aligned} \quad (3.15)$$

Remark 3.1 By observing the result (3.15), it is found that the period of the in-plane motion (x_j, y_j) is determined only by the secular motion of the true anomaly $\bar{f}^t$, while the period of the cross-track motion z_j is determined by both the secular motions of the true anomaly $\bar{f}^t$ and the argument of perigee $\bar{\omega}^t$. Generally, the period difference between the in-plane motion (x_j, y_j) and the cross-track motion z_j may cause the tumbling of the relative orbit. It implies that three conditions in (3.13) are generally not enough to ensure a periodic closed relative orbit.

Remark 3.2 Upon further investigation into the last equation in (3.15), it is found that the cross-track motion is independent of the time-varying variable $\bar{\omega}^t$ if one of the following conditions is satisfied

$$\bar{i} = i_c \quad (3.16)$$

or

$$\bar{i} = 0 \quad (3.17)$$

or

$$\delta \bar{\Omega}_j = 0 \quad (3.18)$$

where $i_c = 63.435^\circ$ or 116.565° is the critical inclination.

From the J_2 perturbation theory, it is known that, under condition (3.16), the apsidal rotation rate is zero and the perigee is frozen at somewhere

$$\dot{\bar{\omega}}^t = 0 \quad \text{and} \quad \bar{\omega}^t = \bar{\omega} \quad \forall \bar{i} = i_c \quad (3.19)$$

Condition (3.16) illustrates that the periodic relative trajectory can be achieved at the critical inclination, which will be discussed in detail in the next section.

Conditions (3.17) and (3.18) together with $\delta \bar{i}_j = 0$ actually imply that the member satellite shares the same orbital plane with the reference satellite so that the relative orbit does not have the cross-track motion and

$$z_j = 0 \quad (3.20)$$

The formations under the conditions (3.17) or (3.18) are referred as the in-plane formation, which are discussed in detail in Sect. 3.4.

3.3 Periodic and Quasi-periodic Relative Orbits at Critical Inclination

Orbit at the critical inclination is a practical option for long-term satellite formation flying around an oblate Earth. This section establishes that passive and periodic relative orbits with cross-track motion are only possible at the critical inclination under conditions that all satellites share the same inclination, the same semimajor axis, and the same eccentricity. Furthermore, a quasi-periodic relative orbit is achieved near the critical inclination by matching the orbital energy and the polar component of the orbital angular momentum. These matching conditions can be expressed in terms of the actual orbit variables. The theoretical result shows that these conditions are equivalent to matching the orbital period and the rate of the mean anomaly.

3.3.1 Periodic Relative Orbit

In this subsection, the conditions of the passive and periodic relative orbit with cross-track motion under J_2 perturbation are presented.

$$z_j = 0 \quad (3.20)$$

By substituting (3.16) and (3.19) into (3.15), the solution of the passive and periodic relative orbit is derived as

$$\begin{aligned}
 x_j(\bar{f}^t) &= \frac{\bar{a} \cdot \bar{e} \delta \bar{M}_j}{\bar{\eta}} \cdot s_{\bar{f}^t} \\
 y_j(\bar{f}^t) &= \frac{\bar{a} \delta \bar{M}_j}{\bar{\eta}} \cdot (1 + \bar{e} c_{\bar{f}^t}) + \frac{\bar{a} \bar{\eta}^2 (c_{i_c} \delta \bar{\Omega}_j + \delta \bar{\omega}_j)}{1 + \bar{e} c_{\bar{f}^t}} \\
 z_j(\bar{f}^t) &= -\bar{a} \bar{\eta}^2 s_{i_c} \delta \bar{\Omega}_j \cdot \frac{c_{(\bar{f}^t + \bar{\omega})}}{1 + \bar{e} c_{\bar{f}^t}}
 \end{aligned} \tag{3.21}$$

The results established are presented in the following remark.

Remark 3.3 It can be concluded that if the J_2 perturbation is present, a *passive and periodic relative orbit* with the cross-track motion requires four conditions

$$\delta \bar{a}_j = 0 \quad \delta \bar{i}_j = 0 \quad \delta \bar{e}_j = 0 \quad \bar{i} = i_c \tag{3.22}$$

i.e., there is one more condition $\bar{i} = i_c$ than the strict J_2 invariant relative orbit conditions (3.13).

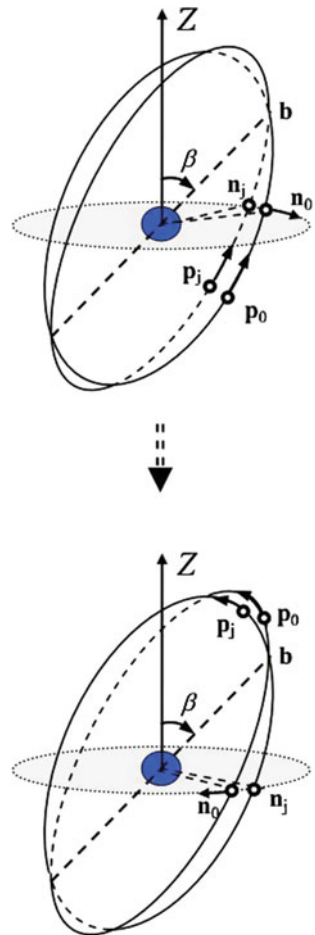
This remark can be illustrated with the aid of Fig. 3.2. Suppose that the strict J_2 invariant relative orbit conditions (3.13) are satisfied and the orbital planes of two satellites do not overlap. It follows that the secular rates of the ascending nodes $(\mathbf{n}_0, \mathbf{n}_j)$ and the perigees $(\mathbf{p}_0, \mathbf{p}_j)$ are the same for both the reference and the member satellite orbital planes. They ensure the angle β between the z -axis and the intersection line $\mathbf{b}$ of two orbital planes is a constant. However, the angles between the perigees $(\mathbf{p}_0, \mathbf{p}_j)$ and the intersection line $\mathbf{b}$ are varying, and hence, it causes the tumbling of the relative orbit. It is at the critical inclination that perigees $(\mathbf{p}_0, \mathbf{p}_j)$ are frozen so that the relationship between two satellite orbits does not change.

Remark 3.4 The above remark reveals that the critical inclination is very important for long-term formation flying. Since the orbits at the critical inclination cover most parts of the Earth surface, these orbits may meet most mission requirement in satellite formation flying.

3.3.2 Quasi-periodic Relative Orbit

The periodic relative orbit conditions (3.22) are very strong and leave only three degrees of freedom to design the formation geometry. In this subsection, one more degrees of freedom is yielded by establishing a set of quasi-periodic relative orbit conditions.

Fig. 3.2 Relative motion affected by the apsidal rotation



It is known that both the 2-body and J_2 gravities are conservative and axial symmetric. So, if only the 2-body and J_2 gravities are considered, a satellite motion will be subject to some conservative quantities. Among possible conservative quantities, two classical mechanical quantities are highlighted, i.e., the orbital energy and the polar component of the orbital angular momentum. Next, these two conservative quantities are shown play an important role in formation flying at the critical inclination.

The difference of energies of the reference and j th member satellites is denoted as

$$\delta E_j = E_j - E \quad (3.23)$$

and the difference of their polar components of the orbital angular momentum is denoted as

$$\delta H_j = H_j - H \quad (3.24)$$

Generally, if the J_2 gravity is included and both the satellites have the same orbital period, the differences (3.23) and (3.24) may not be zero. However, if the reference satellite flies in a orbit at the critical inclination, i.e., $\bar{i} = i_c$, conditions

$$\delta E_j = 0 \quad (3.25)$$

and

$$\delta H_j = 0 \quad (3.26)$$

will guarantee not only the orbital periods but also the mean anomaly rates of the two satellites are identical, which will be proved in the following derivations.

The quantities E and H of the reference satellite are expressed in terms of the mean orbital elements as

$$E = -\frac{\mu}{2\bar{a}} + \frac{J_2\mu R_e^2}{4\bar{a}^3\bar{\eta}^3}(1 - 3c_i^2) \quad (3.27)$$

$$H = c_i\bar{\eta}\sqrt{\mu\bar{a}} \quad (3.28)$$

where μ is the Earth gravitational constant; J_2 is the second zonal harmonic coefficient of the Earth; and R_e is the Earth equatorial radius. The variations of (3.27) and (3.28) are

$$\delta E = \left(\frac{\mu}{2\bar{a}^2} - \frac{3J_2\mu R_e^2}{4\bar{a}^4\bar{\eta}^3}(1 - 3c_i^2) \right) \delta\bar{a} + \frac{3J_2\mu R_e^2}{4\bar{a}^3\bar{\eta}^4} (s_{2\bar{i}}\bar{\eta}\delta\bar{i} - (1 - 3c_i^2)\delta\bar{\eta}) \quad (3.29)$$

$$\delta H = c_i\sqrt{\mu\bar{a}}\delta\bar{\eta} - \bar{\eta}s_i\sqrt{\mu\bar{a}}\delta\bar{i} + \frac{1}{2}\bar{\eta}c_i\sqrt{\frac{\mu}{\bar{a}}}\delta\bar{a} \quad (3.30)$$

Applying condition (3.16), the differences (3.23) and (3.24) of the j th member satellite at the critical inclination are evaluated by the variations (3.29) and (3.30) as

$$\delta E_j = \left(\frac{\mu}{2\bar{a}^2} - \frac{3J_2\mu R_e^2}{10\bar{a}^4\bar{\eta}^3} \right) \delta\bar{a}_j + \frac{3J_2\mu R_e^2}{10\bar{a}^3\bar{\eta}^4} (\pm 2\bar{\eta}\delta i_j - \delta\bar{\eta}_j) \quad (3.31)$$

$$\delta H_j = \pm \sqrt{\frac{\mu\bar{a}}{5}} \left(\frac{\bar{\eta}}{2\bar{a}} \delta\bar{a}_j - (\pm 2\bar{\eta}\delta\bar{i}_j - \delta\bar{\eta}_j) \right) \quad (3.32)$$

where upper and lower signs of $\pm$ are for $\bar{i} = 63.425^\circ$ and $\bar{i} = 116.565^\circ$, respectively. Applying conditions (3.25) and (3.26) and assuming

$$\bar{a}^2 \neq \frac{3J_2 R_e^2}{10\bar{\eta}^3} \quad (3.33)$$

Equations (3.31) and (3.32) yield

$$\delta \bar{a}_j = 0 \quad (3.34)$$

$$\delta \bar{\eta}_j = \pm 2\bar{\eta} \delta \bar{i}_j \quad (3.35)$$

Actually, assumption (3.33) is always satisfied for an Earth orbital satellite. Since the orbit altitude at the perigee is larger than the Earth's radius, we have

$$\bar{a}^2 > \frac{R_e^2}{(1 - \bar{e})^2} \quad (3.36)$$

Substituting

$$\bar{a}^2 = \frac{3J_2 R_e^2}{10\bar{\eta}^3} \quad (3.37)$$

into (3.36) yields that

$$\frac{3J_2}{10} > \frac{\bar{\eta}^3}{(1 - \bar{e})^2} > 1 \quad (3.38)$$

is necessary such that (3.37) is true. It is known that $J_2 = 0.00108$ so that (3.38) does not have a solution. As a result, assumption (3.33) is always true.

This derivation reveals that conditions (3.25) and (3.26) are equivalent to conditions (3.34) and (3.35) at the critical inclination. Using the variation form of the identity (3.7) to replace $\delta \bar{\eta}_j$ in (3.35), an alternate expression of (3.35) is obtained as

$$\delta \bar{i}_j = \mp \frac{\bar{e} \delta \bar{e}_j}{2\bar{\eta}^2} \quad (3.39)$$

Next, it is investigated how conditions (3.34) and (3.35) work on other orbit element differences $(\delta \bar{\Omega}_j^t, \delta \bar{\omega}_j^t, \delta \bar{M}_j^t)$ at the critical inclination. The J_2 perturbation theory proves that

$$\dot{\bar{\Omega}} = -\frac{3}{2} J_2 \frac{R_e^2}{\bar{a}^2} \frac{\bar{n}}{\bar{\eta}^4} c_{\bar{i}} \quad (3.40)$$

$$\dot{\omega} = \frac{3}{4} J_2 \frac{R_e^2}{\bar{a}^2} \frac{\bar{n}}{\bar{\eta}^4} (5c_i^2 - 1) \quad (3.41)$$

$$\dot{M} = \bar{n} - \frac{3}{4} J_2 \frac{R_e^2}{\bar{a}^2} \frac{\bar{n}}{\bar{\eta}^3} (1 - 3c_i^2) \quad (3.42)$$

where

$$\bar{n} = \sqrt{\mu/\bar{a}^3} \quad (3.43)$$

is the mean motion. Substitute (3.43) into (3.40)–(3.42) and then take the variations which give the rates of differential orbit elements as

$$\delta\dot{\Omega} = \frac{3J_2R_e^2\sqrt{\mu}}{4\bar{a}^{9/2}\bar{\eta}^5} (7c_i\bar{\eta}\delta\bar{a} + 2\bar{a}(4c_i\delta\bar{\eta} + s_i\bar{\eta}\delta\bar{i})) \quad (3.44)$$

$$\delta\dot{\omega} = -\frac{3J_2R_e^2\sqrt{\mu}}{8\bar{a}^{9/2}\bar{\eta}^5} \cdot 7(5c_i^2 - 1)\bar{\eta}\delta\bar{a} - \frac{3J_2R_e^2\sqrt{\mu}}{8\bar{a}^{9/2}\bar{\eta}^5} \cdot 2\bar{a}(4(5c_i^2 - 1)\delta\bar{\eta} + 5\bar{\eta}s_{2i}\delta\bar{i}) \quad (3.45)$$

$$\delta\dot{M} = \frac{3\sqrt{\mu}}{2\bar{a}^{5/2}} \left(\frac{7J_2R_e^2}{4\bar{a}^2\bar{\eta}^3} (1 - 3c_i^2) - 1 \right) \delta\bar{a} + \frac{9J_2R_e^2\sqrt{\mu}}{4\bar{a}^{7/2}\bar{\eta}^4} ((1 - 3c_i^2)\delta\bar{\eta} - \bar{\eta}s_{2i}\delta\bar{i}) \quad (3.46)$$

Applying (3.16) and (3.34)–(3.35), the rates of the orbit element differences $(\delta\bar{\Omega}_j^t, \delta\bar{\omega}_j^t, \delta\bar{M}_j^t)$ of the j th member satellite at the critical inclination under conditions (3.25) and (3.26) are derived from (3.44) to (3.46) as

$$\delta\dot{\bar{\Omega}}_j = \frac{3\sqrt{5}J_2R_e^2\sqrt{\mu}}{\bar{a}^{7/2}\bar{\eta}^4} \delta\bar{i}_j \quad (3.47)$$

$$\delta\dot{\bar{\omega}}_j = \mp \frac{3J_2R_e^2\sqrt{\mu}}{\bar{a}^{7/2}\bar{\eta}^4} \delta\bar{i}_j \quad (3.48)$$

$$\delta\dot{\bar{M}}_j = 0 \quad (3.49)$$

Define an angle $\delta\bar{\lambda}_j^t$ as

$$\delta\bar{\lambda}_j^t = \delta\bar{\omega}_j^t + \delta\bar{\Omega}_j^t c_i \quad (3.50)$$

which describes a perturbed motion between satellites caused by the J_2 gravity. Using the commutative principle

$$\frac{d}{dt}(\delta x) = \delta \dot{x} \quad (3.51)$$

Taking time derivative of (3.50) and considering (3.16), (3.47), and (3.48) yield

$$\delta \dot{\lambda}_j = \delta \dot{\omega}_j + \delta \dot{\Omega}_j c_{i_c} = 0 \quad (3.52)$$

By combining (3.49) and (3.52), it is concluded that the orbital rate difference between the reference satellite and the j th member satellite is zero under conditions (3.25) and (3.26) at the critical inclination

$$\delta \dot{M}_j + \delta \dot{\omega}_j + \delta \dot{\Omega}_j c_{i_c} = 0 \quad (3.53)$$

In other words, it implies the two satellites have the same nodal orbital period (Schaub and Junkins 2003)

$$\delta T = 0 \quad (3.54)$$

The remarkable results (3.49) and (3.52) suggest that, under conditions (3.25) and (3.26) at the critical inclination, the time-varying variables $\delta \bar{M}_j^t$ and $\delta \bar{\omega}_j^t + \delta \bar{\Omega}_j^t c_{i_c}$ can be replaced by constants $\delta \bar{M}_j$ and $\delta \bar{\lambda}_j$ as

$$\delta \bar{M}_j^t = \delta \bar{M}_j \quad (3.55)$$

$$\delta \bar{\omega}_j^t + \delta \bar{\Omega}_j^t c_{i_c} = \delta \bar{\lambda}_j \quad (3.56)$$

Now, by substituting (3.16), (3.19), (3.34)–(3.35), (3.39), and (3.55)–(3.56) into (3.12), the solution of the satellite relative orbit under conditions (3.25) and (3.26) near the critical inclination is derived as

$$\begin{aligned} x_j(\bar{f}^t) &= \frac{\bar{a} \cdot \bar{e} \delta \bar{M}_j}{\bar{\eta}} \cdot s_{\bar{f}^t} - \bar{a} \delta \bar{e}_j \cdot c_{\bar{f}^t} \\ y_j(\bar{f}^t) &= \frac{\bar{a} \delta \bar{e}_j (2 + \bar{e} c_{\bar{f}^t}) s_{\bar{f}^t}}{1 + \bar{e} c_{\bar{f}^t}} + \frac{\bar{a} \delta \bar{M}_j (1 + \bar{e} c_{\bar{f}^t})}{\bar{\eta}} + \frac{\bar{a} \bar{\eta}^2 \delta \bar{\lambda}_j}{1 + \bar{e} c_{\bar{f}^t}} \\ z_j(\bar{f}^t, \delta \bar{\Omega}_j^t) &= \frac{-\bar{a}}{1 + \bar{e} c_{\bar{f}^t}} \left(\bar{\eta}^2 s_{i_c} \delta \bar{\Omega}_j^t c_{(\bar{f}^t + \bar{\omega})} \pm \frac{\bar{e} \delta \bar{e}_j s_{(\bar{f}^t + \bar{\omega})}}{2} \right) \end{aligned} \quad (3.57)$$

Remark 3.5 Similar to solution (3.15), the in-plane trajectory (x_j, y_j) of solution (3.57) is periodically closed. Nevertheless, conditions (3.16) and (3.34)–(3.35) for

solution (3.57) are weaker than conditions (3.13) for solution (3.15). They allow four degrees of freedom to design the satellite formation geometry, while conditions (3.13) grant only three degrees of freedom.

Remark 3.6 Conditions (3.25)–(3.26) or (3.34)–(3.35) are referred as the *quasi-periodic relative orbit conditions* near the critical inclination.

3.3.3 *Quasi-periodic Relative Orbit Conditions in Terms of Actual Orbit Variables*

The quasi-periodic relative orbit conditions (3.34) and (3.35) are expressed in the mean orbital elements. They provide insight into the relationships among the mean orbit element differences so as to gain an understanding on the formation geometry. However, they involve a complex transformation to establish the initial conditions of the member satellite in the local frame. Since the orbital energy and the polar component of the orbital angular momentum are the conservative quantities, it is possible to express conditions (3.25) and (3.26) in terms of the actual orbit variables.

There are different approaches to select the actual orbit variables of the reference satellite. A convenient set of five reference satellite variables $(r, \dot{r}, h, \theta, i)$, which are presented in Xu and Wang (2008) and called compact reference satellite variables (CRSV), is used in this book. The physical meanings of CRSV are the radius r , the velocity component along the radial direction $\dot{r}$, the angular momentum magnitude h , the true latitude θ , and the inclination i . On the other hand, the relative orbit variables of the j th member satellite are commonly expressed by the relative position components (x_j, y_j, z_j) and the relative velocity components $(\dot{x}_j, \dot{y}_j, \dot{z}_j)$ in the LVLH frame. Next, the expression of conditions (3.25) and (3.26) in terms of 11 actual orbit variables $(r, \dot{r}, h, \theta, i)$ and $(x_j, y_j, z_j, \dot{x}_j, \dot{y}_j, \dot{z}_j)$ is derived.

In the LVLH frame, the position vectors of the reference and the j th member satellites in the inertial space are

$$\mathbf{r} = r\hat{\mathbf{x}} \quad \mathbf{r}_j = (r + x_j)\hat{\mathbf{x}} + y_j\hat{\mathbf{y}} + z_j\hat{\mathbf{z}} \quad (3.58)$$

Taking time derivative produces their velocity vectors (Xu and Wang 2008) as

$$\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{x}} + \frac{h}{r}\hat{\mathbf{y}} \quad \dot{\mathbf{r}}_j = v_{jx}\hat{\mathbf{x}} + v_{jy}\hat{\mathbf{y}} + v_{jz}\hat{\mathbf{z}} \quad (3.59)$$

with the velocity components along LVLH axes being

$$\begin{aligned}
v_{jx} &= \dot{x}_j + \dot{r} - y_j \omega_z \\
v_{jy} &= \dot{y}_j + (r + x_j) \omega_z - z_j \omega_x \\
v_{jz} &= \dot{z}_j + y_j \omega_x
\end{aligned} \tag{3.60}$$

and the angular velocity components being

$$\omega_x = -\frac{k_{J2} s_{2i} s_\theta}{h r^3} \quad \omega_z = \frac{h}{r^2} \tag{3.61}$$

In (3.61), k_{J2} is a constant defined by

$$k_{J2} = 3J_2 \mu R_e^2 / 2 \tag{3.62}$$

Using the position vectors (3.59) and the velocity vectors (3.60), the polar components of the angular momentum of the reference and the member satellites can be computed by

$$H = (\mathbf{r} \times \dot{\mathbf{r}}) \cdot \hat{\mathbf{Z}} \quad H_j = (\mathbf{r}_j \times \dot{\mathbf{r}}_j) \cdot \hat{\mathbf{Z}} \tag{3.63}$$

where the unit vector $\hat{\mathbf{Z}}$ of the polar axis can be expressed in the LVLH frame as

$$\hat{\mathbf{Z}} = s_\theta s_i \hat{\mathbf{x}} + c_\theta s_i \hat{\mathbf{y}} + c_i \hat{\mathbf{z}} \tag{3.64}$$

Substituting (3.59)–(3.60) and (3.64) into (3.63), then condition (3.26) $\delta H_j = H_j - H = 0$ becomes

$$v_{jx}(z_j c_\theta s_i - y_j c_i) + v_{jy}((x_j + r)c_i - z_j s_\theta s_i) + v_{jz}(y_j s_\theta - (x_j + r)c_\theta)s_i = h c_i \tag{3.65}$$

On the other hand, energy-matching condition (3.25) can be written as

$$K + U = K_j + U_j \tag{3.66}$$

where K and K_j are kinetic energies per unit mass of the two satellites, which are computed by

$$K = \frac{1}{2} \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} \quad K_j = \frac{1}{2} \dot{\mathbf{r}}_j \cdot \dot{\mathbf{r}}_j \tag{3.67}$$

U and U_j are potential energies per unit mass of the two satellites, which are derived in Xu and Wang (2008) as

$$\begin{aligned}
 U &= -\frac{\mu}{r} - \frac{k_{J2}}{3r^3} + \frac{k_{J2}s_\theta^2 s_i^2}{r^3} \\
 U_j &= -\frac{\mu}{r_j} - \frac{k_{J2}}{3r_j^3} + \frac{k_{J2}r_{jZ}^2}{r_j^5}
 \end{aligned} \tag{3.68}$$

with

$$\begin{aligned}
 r_j &= \sqrt{(r+x_j)^2 + y_j^2 + z_j^2} \\
 r_{jZ} &= (r+x_j)s_i s_\theta + y_j s_i c_\theta + z_j c_i
 \end{aligned} \tag{3.69}$$

Substituting (3.67) and (3.68) into (3.66) and considering (3.59), condition (3.25) becomes

$$v_{jx}^2 + v_{jy}^2 + v_{jz}^2 = \dot{r}^2 + \frac{h^2}{r^2} + 2\mu\left(\frac{1}{r_j} - \frac{1}{r}\right) + 2k_{J2}\left(\frac{1}{3r_j^3} - \frac{1}{3r^3} - \frac{r_{jZ}^2}{r_j^5} + \frac{s_\theta^2 s_i^2}{r^3}\right) \tag{3.70}$$

Remark 3.7 Now, the quasi-periodic relative orbit conditions (3.25) and (3.26) are expressed in terms of the actual orbit variables in Eqs. (3.65) and (3.70).

3.3.4 Numerical Simulations

Numerical simulations are performed to verify the proposed quasi-periodic relative orbit conditions. In the simulations, a reference satellite flies in an elliptic orbit at the critical inclination, while a member satellite flies around the reference satellite. Three cases are simulated and compared. To evaluate the different simulated cases, the same reference satellite orbit is used for each simulation, which is represented by the mean orbit elements as follows:

$$\begin{aligned}
 \bar{a} &= 8000 \text{ km} & \bar{e} &= 0.1 & \bar{i} &= 63.4349^\circ \\
 \bar{\omega} &= 45^\circ & \bar{\Omega} &= 0 & \bar{f} &= 0
 \end{aligned} \tag{3.71}$$

Care must be taken when setting up the initial conditions of the simulation so that the mean orbit elements (3.71) are converted to the osculating orbit elements first and then transformed to the CRSV. On the other hand, the same actual initial relative position of the member satellite is assigned for each case

$$x_j(0) = 0.5 \text{ km} \quad y_j(0) = 0 \quad z_j(0) = 1 \text{ km} \tag{3.72}$$

but the actual initial relative velocity is adjusted by three different techniques for the periodic relative orbit. In the numerical simulations, both the actual initial conditions of the reference satellite and the member satellite are applied to an exact propagator of the satellite relative motion which accommodates eccentricity, non-linearity, and J_2 perturbation. Next, the details of how the initial velocity of the member satellite is calculated by three different techniques are illustrated.

In the first case, the initial velocity is calculated by the technique (Inalhan et al. 2002) for the linear relative dynamics in the elliptic orbit as

$$\dot{x}_j(0) = 0 \quad \dot{z}_j(0) = 0 \quad (3.73)$$

$$\dot{y}(0) = \frac{-\bar{n}(2 + \bar{e})x(0)}{(1 + \bar{e})^{1/2}(1 - \bar{e})^{3/2}} \quad (3.74)$$

This initial velocity may produce a periodic relative orbit if the J_2 perturbation and the nonlinearity are ignored.

In the second case, the initial velocity is calculated by the energy-matching technique (Gurfil 2005; Xing et al. 2007) for the classical elliptic Keplerian motion. Such energy-matching condition is equivalent to condition (3.70) by canceling J_2 effect (setting $J_2 = 0$), which is explicitly expressed as

$$(\dot{x}_j + \dot{r} - y_j\omega_z)^2 + (\dot{y}_j + r\omega_z + x_j\omega_z)^2 + \dot{z}_j^2 = \dot{r}^2 + r^2\omega_z^2 + 2\mu\left(\frac{1}{r_j} - \frac{1}{r}\right) \quad (3.75)$$

In the simulation of the second case, instead of (3.74), initial velocity component $\dot{y}(0)$ is solved from condition (3.75), while initial velocity components $(\dot{x}_j(0), \dot{z}_j(0))$ remain the same as (3.73). As explained earlier, this initial velocity may produce a periodic relative orbit if the J_2 perturbation is ignored.

In the third case, the newly developed quasi-periodic relative orbit conditions (3.65) and (3.70) are applied to calculate the actual initial velocity of the member satellite. Since the two conditions produce two velocity components, $\dot{x}_j(0) = 0$ is kept and let $(\dot{y}_j(0), \dot{z}_j(0))$ be determined by the algebraic equations (3.65) and (3.70).

Table 3.1 shows the calculated values of the initial velocity with the three different techniques.

Simulations are performed for each case, and the results of 96-h formation flying are shown in Figs. 3.3, 3.4, and 3.5. It can be seen that both the first case and the second case have a fast drift in the along-track (y) direction. The drift in Fig. 3.3 is

Table 3.1 Initial velocities (m/s)

Case	$\dot{x}_j(0)$	$\dot{y}_j(0)$	$\dot{z}_j(0)$
(1) Linear w/o J_2	0	1.03415	0
(2) Nonlinear w/o J_2	0	-1.03486	0
(3) Quasi-periodic	0	-1.03564	-1.05058

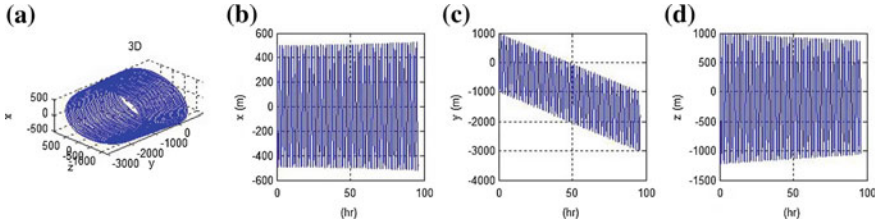


Fig. 3.3 Under linear condition

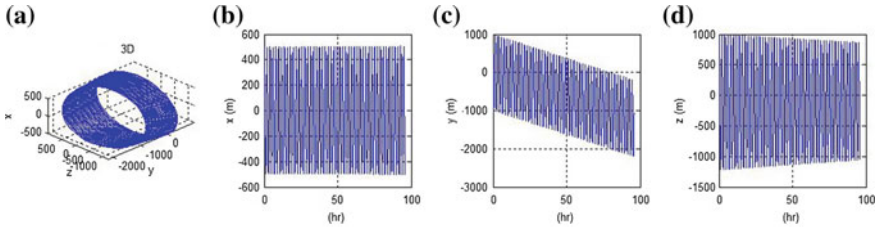


Fig. 3.4 Under nonlinear condition without J_2 gravity

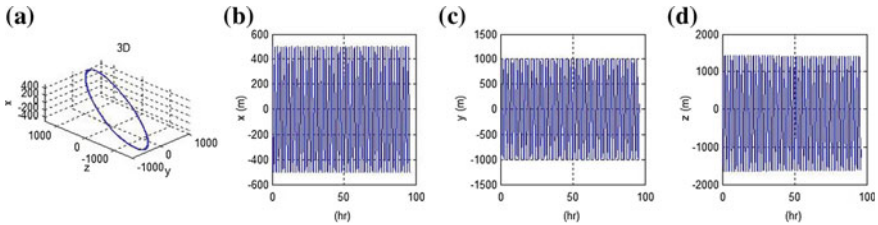


Fig. 3.5 Under quasi-periodic condition

about 2 km which is caused by the ignored nonlinearity and the J_2 perturbation. The drift in Fig. 3.4 is about 1.2 km which is caused solely by the J_2 perturbation. This result verifies the claim made by Schaub and Alfriend (2001) that the orbital energies of different orbits with the same orbital period may not be the same when the J_2 perturbation is present. The drift magnitudes of the first two cases show that the J_2 perturbation and the ignored nonlinearity induce large drift errors in the same order. Meanwhile, it is observed there is a slow reduction in the cross-track (z) direction for both the first and the second cases. This reduction is generated from the tumbling of the relative orbit which is caused by the J_2 perturbation. On the other hand, the result of the third case (as shown in Fig. 3.5) is nearly a perfect periodic relative orbit. Both the along-track drift and the tumbling of the relative orbit are eliminated.

Remark 3.8 As analyzed above, the simulation results actually demonstrate the separate effect of the nonlinearity and the J_2 perturbation. The along-track drift is caused by the ignored nonlinearity and the J_2 perturbation, while the tumbling of the relative orbit is caused by the J_2 perturbation only. If a designed formation does not address the nonlinearity or the J_2 perturbation, then active control must be employed to compensate the formation drift and tumbling. Since to correct the tumbling of the relative orbit requires the maintenance of the perigee and/or the right ascension of the ascending node, the fuel consumption to compensate the J_2 perturbation is expected to be much more than that for the nonlinearity (at least for formations about 2-km separation) which involves the orbital period only. Hence, it is believed that the nonlinearity and especially the J_2 perturbation effects are not negligible in the design of long-term satellite formation.

Remark 3.9 In summary, neither the linear periodic condition (3.74) nor the energy-matching condition (3.75) can establish a periodic relative orbit when the J_2 perturbation is present. The quasi-periodic relative orbit is achieved in the critical inclination by using the proposed conditions (3.65) and (3.70). It is a promising solution to design satellite formations with cross-track motion.

3.4 In-Plane Satellite Formation in Eccentric Orbits

In this section, the passive and periodic in-plane formation is investigated. It is proved that a passive in-plane formation is periodic and stable in J_2 perturbed eccentric orbits by setting all satellites to have the same mean orbit elements of semimajor axis, eccentricity, inclination, and right ascension of ascending node. As a result, the geometry design approach of some special cases will be proposed. In particular, a special in-plane formation named as Almost Constant Separation Formation is achieved by suitably designing the differential secular elements of the argument of perigee and the mean anomaly. Further analysis shows that the separation distances between satellites in this special in-plane formation can be specified at will and are almost invariant if the eccentricity is smaller than 0.3.

Condition (3.17) or condition (3.18) together with $\delta \bar{r}_j = 0$ actually confines the member satellite in the same orbital plane of the reference satellite. As a generalization to the study (Gurfil 2005) for circular orbits, the formations under the condition (3.17) or (3.18) and under the strict J_2 invariant relative conditions in (3.13) are referred as the *in-plane formation*.

Since the orbit elements used here are singular in zero inclination condition (3.17), without loss of generality, the relative trajectory of an in-plane formation can be studied by substituting (3.18) into (3.15)

$$x_j(\bar{r}^t) = \frac{\bar{a} \cdot \bar{e} \delta \bar{M}_j}{\bar{\eta}} \cdot s_{\bar{r}} \quad (3.76)$$

$$y_j(\bar{f}^t) = \frac{\bar{a}\delta\bar{M}_j}{\bar{\eta}}(1 + \bar{e}c_{\bar{f}^t}) + \frac{\bar{a}\bar{\eta}^2\delta\bar{\omega}_j}{1 + \bar{e}c_{\bar{f}^t}} \quad (3.77)$$

$$z_j = 0 \quad (3.78)$$

It is clear that the relative trajectory of (3.76)–(3.78) is periodic and depends on the sole variable $\bar{f}^t$. So, this in-plane formation is stable under the J_2 perturbation.

As derived in Sect. 3.2, suppose the reference orbit is specified such that four differential orbit elements of $(\delta\bar{a}_j, \delta\bar{l}_j, \delta\bar{e}_j, \delta\bar{\Omega}_j)$ are zero

$$\delta\bar{a}_j = 0 \quad \delta\bar{l}_j = 0 \quad \delta\bar{e}_j = 0 \quad \delta\bar{\Omega}_j = 0 \quad (3.79)$$

The relative trajectory is confined in the same orbit plane as described in (3.76)–(3.78), which is dominated by only two constant parameters $(\delta\bar{M}_j, \delta\bar{\omega}_j)$. Next, the geometry design of the in-plane formation will be discussed in detail by specifying two parameters $(\delta\bar{M}_j, \delta\bar{\omega}_j)$. Since there is no out-plane motion ($z_j = 0$), only the in-plane motion (x_j, y_j) will be investigated.

3.4.1 Identical Anomaly In-Plane Formation

The first in-plane formation is obtained by the simple idea of setting $\delta\bar{M}_j = 0$. As a result, the in-plane motion of (3.76) and (3.77) is simplified to

$$x_j = 0 \quad y_j(\bar{f}^t) = \frac{\bar{a}\bar{\eta}^2\delta\bar{\omega}_j}{1 + \bar{e}c_{\bar{f}^t}} \quad (3.80)$$

This special formation was first discussed by Lane and Axelrad (2006) and referred to as the in-track formation, which conflicts with another formation proposed earlier by Sabol et al. (Gurfil 2005). Since $\delta\bar{M}_j = 0$, the member satellite has the identical mean anomaly as the reference satellite. For this reason, this formation is named as the *identical anomaly in-plane formation (IAIF)*. An obvious property of this formation is that all satellites in the formation have the same altitude ($x_j = 0$) above the Earth surface.

The distance between the member satellite and the reference satellite is

$$\rho_j = |y_j| = \frac{\bar{a}(1 - \bar{e}^2)|\delta\bar{\omega}_j|}{1 + \bar{e}c_{\bar{f}^t}} \quad (3.81)$$

Table 3.2 Intersatellite distance variation of IAIF

$\bar{e}$	0	0.1	0.2	0.3
$\Delta(\bar{e})$	1.000	0.818	0.667	0.538
%	0	19.2 %	33.3 %	46.2 %

Equation (3.81) illustrates that satellites never collide ($\rho_j > 0$) in an eccentric orbit ($\bar{e} \neq 0$) as long as ($\delta\bar{\omega}_j \neq 0$). Actually, the condition $\delta\bar{\omega}_j = 0$ in this formation means that the member satellite occupies the same position as the reference satellite.

The intersatellite distance ρ_j in (3.81) achieves the maximum at the apogee ($\bar{f}^t = \pi$) as

$$\max(\rho_j) = \bar{a}(1 + \bar{e})|\delta\bar{\omega}_j| \quad (3.82)$$

and attains the minimum at the perigee ($\bar{f}^t = 0$) as

$$\min(\rho_j) = \bar{a}(1 - \bar{e})|\delta\bar{\omega}_j| \quad (3.83)$$

The ratio of the minimum to the maximum is

$$\Delta(\bar{e}) = \frac{\min \rho_j}{\max \rho_j} = \frac{1 - \bar{e}}{1 + \bar{e}} \quad (3.84)$$

It is clear that the ratio $\Delta(\bar{e})$ solely depends on the eccentricity $\bar{e}$ and is presented in Table 3.2 for $\bar{e} \in [0, 0.3]$.

It is found that the intersatellite distance ρ_j changes a lot in an eccentric orbit ($\bar{e} \neq 0$). As shown in Table 3.2, the intersatellite distance may change 19.2 % even if the eccentricity is 0.1.

3.4.2 Differential Anomaly In-Plane Formation

In this subsection, special attention is paid to investigate another formation whose member satellite has different mean anomaly, i.e., $\delta\bar{M}_j \neq 0$. In contrast to IAIF, this formation is named as the *differential anomaly in-plane formation (DAIF)*.

To facilitate the relative trajectory analysis of the DAIF formation, a transformation for the differential orbit elements ($\delta\bar{M}_j, \delta\bar{\omega}_j$) is introduced as

$$p_j = \frac{\bar{\eta}\delta\bar{\omega}_j}{\delta\bar{M}_j} \quad q_j = \frac{\bar{a}\delta\bar{M}_j}{\bar{\eta}} \quad (3.85)$$

Because of $\delta\bar{M}_j \neq 0$, the range of (p_j, q_j) is

$$p_j \in (-\infty, \infty) \quad q_j \neq 0 \quad (3.86)$$

Inserting (3.85) into (3.76) and (3.77), the relative trajectory of a DAIF is converted to

$$x_j(\bar{f}^t) = q_j \bar{e} \cdot s_{\bar{f}^t} \quad (3.87)$$

$$y_j(\bar{f}^t) = q_j \left(1 + \bar{e} c_{\bar{f}^t} + \frac{p_j \bar{\eta}^2}{1 + \bar{e} c_{\bar{f}^t}} \right) \quad (3.88)$$

Observing the relative trajectory (3.87) and (3.88), it is immediately found that for a DAIF formation, the direction and size of the relative trajectory are dominated by parameter q_j (corresponding sign and magnitude of $\delta \bar{M}_j$), while the shape of the relative trajectory is dominated by parameter p_j (corresponding ratio of $\delta \bar{\omega}_j$ to $\delta \bar{M}_j$).

Some further insights of the relative trajectory (3.87) and (3.88) are disclosed by the following properties.

Property 3.1 *Relative trajectory (3.87) and (3.88) is periodic and its period is the anomalistic period 2π of $\bar{f}^t$. Moreover, it is symmetric with respect to the axis aligning the along-tracking direction.*

Proof It is clear that the relative trajectory of (3.87) and (3.88) satisfies

$$x_j(\bar{f}^t) = x_j(2\pi + \bar{f}^t) \quad y_j(\bar{f}^t) = y_j(2\pi + \bar{f}^t) \quad (3.89)$$

So, it is periodic and its period is the anomalistic period of $\bar{f}^t$. Furthermore, the relative trajectory also satisfies

$$x_j(\bar{f}^t) = -x_j(-\bar{f}^t) \quad y_j(\bar{f}^t) = y_j(-\bar{f}^t) \quad (3.90)$$

So, it is symmetric with respect to y-axis, i.e., the axis aligning the along-track direction.

Property 3.1 suggests that the relative trajectory can be studied within one period of

$$\bar{f}^t \in (-\pi, \pi] \quad (3.91)$$

It is noticed that the relative motion x_j in the radial direction is rather simple and just a function of sine. So, in the region of (3.91), Property 3.2 is straightforward.

Property 3.2 *Radial motion (3.87) crosses the zero twice in one period if and only if the reference satellite is at the apsides*

$$\bar{f}^t = 0 \quad \text{or} \quad \bar{f}^t = \pi \quad (3.92)$$

and reaches its extrema if and only if the reference satellite is at one of semilatus rectums. The extrema are

$$\min(y_j) = -\bar{e}|q_j| \quad \max(y_j) = \bar{e}|q_j| \quad (3.93)$$

at $\bar{f}^t = -\pi/2$ and $\bar{f}^t = \pi/2$, respectively. $\square$

Comparing with the relative motion in the radial direction, the relative motion y_j in the along-track direction is a bit complicated. A property of the along-track motion is revealed by Property 3.3.

Property 3.3 *Under the condition*

$$p_j \geq \frac{1+\bar{e}}{1-\bar{e}} \quad \text{or} \quad p_j \leq \frac{1-\bar{e}}{1+\bar{e}} \quad (3.94)$$

the along-track motion (3.88) reaches its extrema twice in one period, which happen when the reference satellite is at the apsides (3.92) and the two extrema are

$$\text{extremum}_1(y_j) = q_j(1 + \bar{e} + p_j(1 - \bar{e})) \quad (3.95)$$

$$\text{extremum}_2(y_j) = q_j(1 - \bar{e} + p_j(1 + \bar{e})) \quad (3.96)$$

On the other hand, under the condition

$$\frac{1-\bar{e}}{1+\bar{e}} < p_j < \frac{1+\bar{e}}{1-\bar{e}} \quad (3.97)$$

(it implies $\bar{e} \neq 0$) the along-track motion (3.88) reaches its extrema four times, which happen when the reference satellite is at the apsides (3.92) and at

$$\bar{f}^t = \pm \cos^{-1} \frac{\bar{\eta}\sqrt{p_j} - 1}{\bar{e}} \quad (3.98)$$

the four extrema are (3.95), (3.96) and

$$\text{extremum}_3(y_j) = \text{extremum}_4(y_j) = 2q_j\bar{\eta}\sqrt{p_j} \quad (3.99)$$

Proof First of all, $\bar{e} = 0$ implies $y_j \equiv q_j(1 + p_j)$. Extrema (3.95), (3.96) are the same at $q_j(1 + p_j)$ and hold in the whole trajectory. Next, the case of $\bar{e} \neq 0$ is discussed.

Taking derivative of (3.88) with respect to $\bar{f}^t$ yields

$$y_j' = \bar{e}q_j s_{\bar{f}^t} \left(\frac{\bar{\eta}^2 p_j}{(1 + \bar{e}c_{\bar{f}^t})^2} - 1 \right) \quad (3.100)$$

Condition (3.94) implies

$$\frac{\bar{\eta}^2 p_j}{(1 + \bar{e}c_{\bar{f}^t})^2} - 1 \neq 0 \quad (3.101)$$

So, $y_j' = 0$ is equivalent to $s_{\bar{f}^t} = 0$, and hence, y_j attains its extrema when the reference satellite is at the apsides (3.92). Substituting $s_{\bar{f}^t} = 0$ into (3.88), extrema (3.95) and (3.96) are solved.

On the other hand, under condition (3.97), $y_j' = 0$ if either $s_{\bar{f}^t} = 0$ or

$$\frac{\bar{\eta}^2 p_j}{(1 + \bar{e}c_{\bar{f}^t})^2} - 1 = 0 \quad (3.102)$$

Equation (3.102) is equivalent to (3.98). So, y_j attains the extrema when the reference satellite is at the apsides (3.92) and at (3.98). Substituting (3.98) into (3.88), another two extrema are solved as (3.99). $\square$

Property 3.3 tells us that the along-track motion may have two or four extrema. Among them, two extrema of (3.99) may vanish in regions (3.94), while another two extrema of (3.95) and (3.96) are always established. Combining Properties 3.2 and 3.3, it is known that the nonvanishing extrema are achieved just when the radial motion crosses the zero and the reference satellite is at the apsides. As a result, the two nonvanishing extrema are more significant. The bias of member satellite is defined as the mean value of two nonvanishing extrema as

$$b_j = \frac{1}{2} (\text{extremum}_1(y_j) + \text{extremum}_2(y_j)) \quad (3.103)$$

Substituting (3.95) and (3.96) into (3.103), the bias is solved to

$$b_j = q_j(1 + p_j) \quad (3.104)$$

Actually, bias (3.103) or (3.104) also coincides with another comprehensive expression of the along-track motion. Let $\bar{E}^t$ be the secular eccentric anomaly as in the approach of the unperturbed Keplerian motion. The following identity (Schaub and Junkins 2003) is true,

$$\frac{\bar{\eta}^2}{1 + \bar{e}c_{\bar{f}^t}} = 1 - \bar{e}c_{\bar{E}^t} \quad (3.105)$$

Using (3.105), the along-track motion (3.88) can be converted to

$$y_j = q_j \bar{e}(c_{\bar{f}^t} - p_j c_{\bar{E}^t}) + q_j(1 + p_j) \quad (3.106)$$

On the right side of Eq. (3.106), the first term is periodic and the second term plays the role of bias, which is the same as expression (3.104). The expression of bias (3.104) illustrates that parameter p_j dominates not only the shape of the relative trajectory but also its bias. It means that if the parameter p_j is unsuitably selected, the relative trajectory of the member satellite may cross the origin or, in other words, collides with the reference satellite. This is demonstrated by the following property.

Property 3.4 *In a DAIF formation, the member satellite may collide with the reference satellite at the apsides if*

$$p_j = -\frac{1 + \bar{e}}{1 - \bar{e}} \quad \text{or} \quad p_j = -\frac{1 - \bar{e}}{1 + \bar{e}} \quad (3.107)$$

Proof The collision happens when $x_j = y_j = z_j = 0$ at the same time. Here, $z_j = 0$ always holds. From (3.87), it is known that $x_j = 0$ if

$$s_{\bar{f}^t} = 0 \quad \text{or} \quad \bar{e} = 0 \quad (3.108)$$

The first case of $s_{\bar{f}^t} = 0$ implies

$$\bar{f}^t = 0 \quad \text{or} \quad \bar{f}^t = \pi \quad (3.109)$$

Then, substituting (3.109) into (3.88) and setting $y_j = 0$ yield p_j being one of values in (3.107).

On the other hand, $\bar{e} = 0$ implies that $y_j = 0$ is equivalent to

$$p_j = -1 \quad (3.110)$$

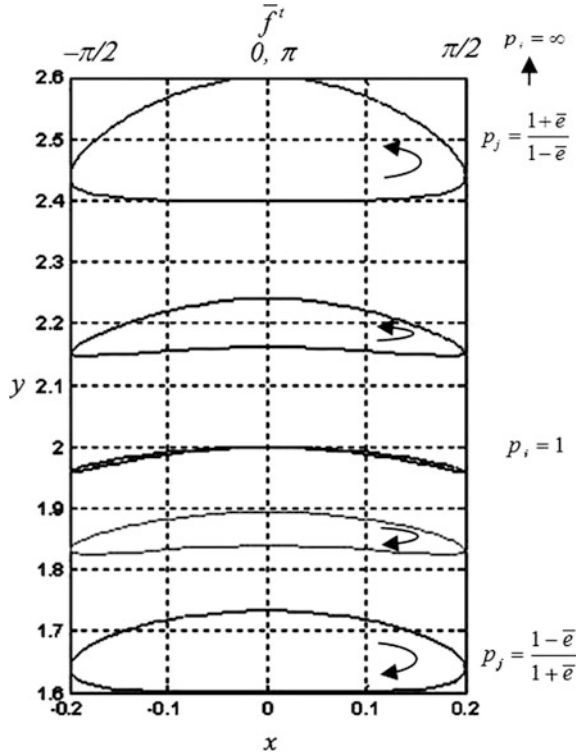
It is a special case of (3.107). □

Properties 3.1–3.4 sketch the essential geometric insights of the relative trajectory of a DAIF formation. They are further illustrated by Figs. 3.6 and 3.7

Setting $\bar{e} = 0.2$ and $q_j = 1$, the relative trajectories of a DAIF formation are shown in Fig. 3.6 for p_j in region

$$\frac{1 - \bar{e}}{1 + \bar{e}} \leq p_j \leq \frac{1 + \bar{e}}{1 - \bar{e}} \quad (3.111)$$

Fig. 3.6 Along-track motion may have four extrema



and shown in Fig. 3.7 for p_j in region

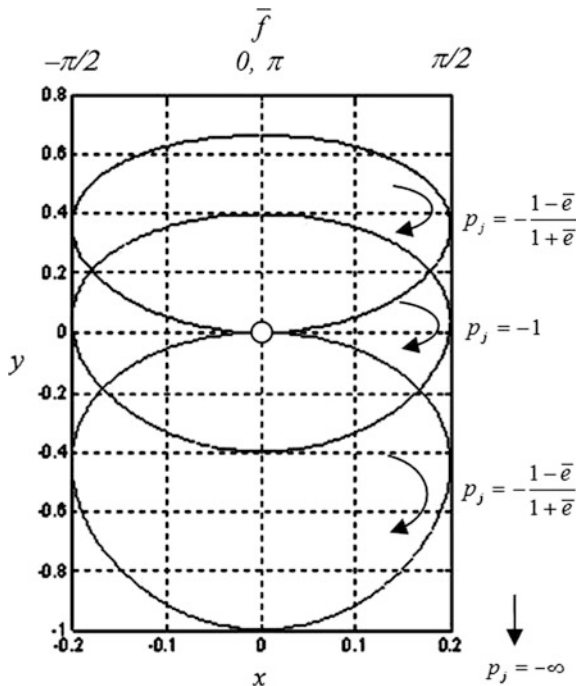
$$-\frac{1+\bar{e}}{1-\bar{e}} \leq p_j \leq -\frac{1-\bar{e}}{1+\bar{e}} \quad (3.112)$$

Figures 3.6 and 3.7 clearly show that parameter p_j plays a very important role in the behavior of the satellite relative motion. Among the whole value space of parameter p_j , two special values should be highlighted.

One special value is $p_j = -1$, where the bias of the along-track motion vanishes (as shown in Fig. 3.7). In this case, the member satellite flies around the reference satellite. However, when the eccentricity approaches zero, the relative trajectory shrinks to the origin and disappears. Moreover, care must be taken to avoid the collision between the member satellite and the reference satellite when p_j changes a little as discussed in Property 3.4.

Another special value is $p_j = 1$, where two nonvanishing extrema of the along-track motion intersect at the same relative position. This result can be concluded from (3.95) and (3.96) by setting $p_j = 1$,

Fig. 3.7 Collisions and zero bias



$$\text{extremum}_1(y_j) = \text{extremum}_2(y_j) = 2q_j \quad (3.113)$$

In depth, the two nonvanishing extrema exchange their relative position around $p_j = 1$. When $p_j < 1$, the first extreme (at perigee) is farther than the second extreme (at apogee) from the origin. When $p_j > 1$, the second extreme (at apogee) exceeds the first extreme (at perigee) and becomes farther from the origin.

Based on the above analysis, it is noticed that the special value of $p_j = 1$ is very significant. As the two nonvanishing extrema are equal, it is expected that the distance from every point on the relative trajectory to the reference satellite changes in a small range. Moreover, Eq. (3.113) shows that when $p_j = 1$ is fixed, the bias can be designed at will by parameter q_j . In the next subsection, the special DAIF formation for $p_j = 1$ is discussed in detail.

3.4.3 Almost Constant Separation Formation

It is known that the two nonvanishing extrema in the along-track direction in the special DAIF for $p_j = 1$ are the same. However, there are two more additional extrema in the along-track direction in this case as revealed in Property 3.3. So, the distance between the member satellite and the reference satellite is not constant. Now, the range of the distance varying is concerned.

Considering (3.87)–(3.88) and (3.78), the distance between the member satellite and the reference satellite is

$$\rho_j(\bar{f}^t) = \sqrt{x_j^2 + y_j^2 + z_j^2} = |q_j| \sqrt{1 + \bar{e}^2 + 2\bar{e}c_{\bar{f}^t} + 2\bar{\eta}^2 + \frac{\bar{\eta}^4}{(1 + \bar{e}c_{\bar{f}^t})^2}} \quad (3.114)$$

Taking derivative with respect to $\bar{f}^t$ yields

$$\rho'_j = \frac{d\rho_j}{d\bar{f}^t} = \frac{q_j^2 \bar{e} s_{\bar{f}^t}}{\rho_j} \left(\frac{\bar{\eta}^4}{(1 + \bar{e}c_{\bar{f}^t})^3} - 1 \right) \quad (3.115)$$

The extrema of distance ρ_j happen when $\rho'_j = 0$, which gives

$$s_{\bar{f}^t} = 0 \quad \text{or} \quad 1 + \bar{e}c_{\bar{f}^t} = \bar{\eta}^{4/3} \quad (3.116)$$

Substituting (3.116) into (3.114), the extrema of distance ρ_j are solved as

$$\max \rho_j = 2|q_j| \quad (3.117)$$

And

$$\min \rho_j = |q_j| \sqrt{\bar{\eta}^2 + 3\bar{\eta}^{4/3}} \quad (3.118)$$

So, the ratio of the minimum to the maximum can be calculated by

$$\Delta(\bar{e}) = \frac{\min \rho_j}{\max \rho_j} = \frac{\sqrt{\bar{\eta}^2 + 3\bar{\eta}^{4/3}}}{2} \quad (3.119)$$

It is clear that the ratio $\Delta(\bar{e})$ solely depends on the eccentricity $\bar{e}$ and is presented in Table 3.3 for $\bar{e} \in [0, 0.3]$. It is found that although the distance between the member satellite and the reference satellite is variable in this special DAIF with $p_j = 1$, the variation is very small if the eccentricity is smaller than 0.3. For this reason, this special formation is named as the *Almost Constant Separation Formation (ACSF)*.

Now, the formation design approach of the ACSF can be presented by combining the periodic constraints (3.79), the geometric constraints $p_j = 1$, the bias (3.104), and the transformation (3.85) as

Table 3.3 Intersatellite distance variation of ACSF

$\bar{e}$	0	0.1	0.2	0.3
$\Delta(\bar{e})$	1.000	0.996	0.985	0.965
%	0	0.4 %	1.5 %	3.5 %

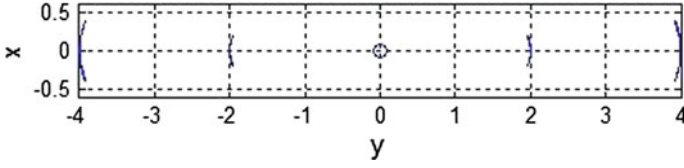


Fig. 3.8 An ACSF of five satellites

$$\begin{aligned} \delta \bar{a}_j &= 0 & \delta \bar{i}_j &= 0 & \delta \bar{e}_j &= 0 \\ \delta \bar{\Omega}_j &= 0 & \delta \bar{\omega}_j &= \frac{b_j}{2a} & \delta \bar{M}_j &= \frac{\bar{\eta} b_j}{2a} \end{aligned} \quad (3.120)$$

The only design parameter is the bias b_j , which specifies the separation distances between the member satellite and the reference satellite. Figure 3.8 shows an example of a 5-satellite ACSF. In this example, the eccentricity is $\bar{e} = 0.2$, and the separation distance between satellites is 2 km. So, the biases of each satellite are specified as

$$b_0 = 0 \quad b_1 = -2 \quad b_2 = -4 \quad b_3 = 2 \quad b_4 = 4 \quad (3.121)$$

Remark 3.10 When the bias b_j of each satellite is specified for ACSF formation, the satellite relative trajectories only depend on the eccentricity and the true anomaly $(\bar{e}, \bar{f}^t)$ but are independent of other orbit elements $(\bar{a}, \bar{i}, \bar{\Omega}^t, \bar{\omega}^t)$ of the reference orbit.

3.5 Conclusions

In this chapter, general conditions are established for the PPRT under the J_2 perturbation in elliptic orbits. There are only two classes of the PPRT. In the first class, the reference satellite orbit is subjected to one constraint: at the critical inclination, while the member satellite orbit must satisfy three constraints: the same semimajor axis, inclination, and eccentricity with the reference orbit. Conditions of the quasi-periodic relative orbit with cross-track motion are also proposed. Such a relative orbit can be achieved by matching the orbital energy and the polar component of the orbital angular momentum for orbits near the critical inclination. As orbits at the critical inclination may cover most parts of the Earth surface and may satisfy most missions for formation flying, and also because the conditions for quasi-periodic relative orbit impose fewer constraints and can be presented in terms of the actual orbit variables, the quasi-periodic relative orbits are good candidates to realize a practical long-term flying satellite formation.

In the second class, the member satellite orbit has to meet four constraints: the same semimajor axis, inclination, eccentricity, and ascending node with the reference orbit. A special satellite formation based on the second class PPRT is defined as the Almost Constant Separation Formation. This special formation introduces one additional geometric constraint for each member satellite: the ratio of the differential mean anomaly to the differential argument of perigee being a constant value associated with the eccentricity. An approach is proposed to design the remaining one-dimensional constraint space via specifying the intersatellite distance. This formation has a simple and almost constant geometry and it is applicable at arbitrary inclination, so it is suitable for most satellite formation flying missions, especially in orbits other than the critical inclination.

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Chapter 4

Nonlinear Optimization of Low-Thrust Trajectory for Satellite Formation

This chapter presents a method to determine fuel-optimal trajectories for satellite formation maneuver using low-thrust continuous propulsion. It is cast as an optimization problem with a desired final satellite formation configuration subject to collision avoidance constraints on the paths of the chief and all deputy satellites, which are correspondingly referred to as reference and member satellites in Chap. 3. The satellite terminal orbit states corresponding to this desired formation configuration are ensured by imposing energy-matching condition and final geometry configuration constraints in the problem formulation. In addition, the developed exact nonlinear relative satellite motion dynamics considering eccentricity and Earth's oblateness is adopted to accurately describe relative satellite orbit. The resulting nonlinear optimal control problem is transcribed into a nonlinear programming problem by the application of a direct transcription method called Legendre pseudospectral method. The nonlinear programming problem is then solved by using sparse nonlinear optimization software named TOMLAB/SNOPT (Holmström et al. 2008). Examples are presented across a range of applications. These examples demonstrate the efficiency of the proposed method in designing fuel-optimal maneuvers for a wide class of satellite formation problems.

4.1 Introduction

Trajectory optimization for formation maneuver is critical to the successful implementation of formation establishment, formation reconfiguration, and replacement of a failed satellite. According to the type of thruster used, trajectory optimization techniques for satellite formation can be roughly categorized into two main approaches: impulsive control, relying on chemical thrusters, and continuous low-thrust control, using electric propulsion system. Impulsive control methods for formation reconfiguration and establishment are proposed in Schaub and Alfriend (2001), Vaddi et al. (2005), Gurfil (2005), and Jifuku et al. (2011). Continuous

low-thrust control methods are actively investigated for satellite formation control. The main advantages are that low-thrust engines (such as ion thruster and plasma thruster) can generate precise thrust output and consume less propellant due to its high specific impulse. Recent advances in the field of electric propulsion have made low-thrust propulsion an operational reality (Polzin 2011; Pillet et al. 2006; Pencil et al. 2004). Literature survey on trajectory optimization methods using continuous low-thrust propulsion is given as follows.

Linear optimization methods have been studied in Tillerson et al. (2002), Richards et al. (2002), Acikmese et al. (2006), Campbell (2002), Zanon and Campbell (2006), Cho et al. (2012), Lee and Park (2011), and Li and Xi (2012) for satellite formation maneuver using continuous low-thrust propulsion. Tillerson et al. (2002) proposed fuel-/time-optimal control algorithm for formation reconfiguration by using linear time-varying relative model. The trajectory optimization algorithms were based on the solutions of linear programming problems. The result was a very flexible optimization framework that can be used offline to analyze various aspects of the mission design and in real time as part of an onboard autonomous formation flying control system. During the satellite formation maneuver, the relative distance between satellites may range from thousands of meters to several meters, so collision avoidance should be considered in the trajectory optimization methods. In order to guarantee safe maneuvering among satellites, Richards et al. (2002) extended the above method to include collision avoidance by using mixed-integer linear programming. The necessary logical constraints for collision avoidance are appended to a fuel-optimizing linear programming by including binary variables in the optimization. The resulting mixed-integer linear program (MILP) problem can be solved using available software. Acikmese et al. (2006) presented a convex guidance algorithm for optimal formation reconfiguration with collision avoidance using CW equations. In this algorithm, the collision avoidance constraints are imposed via separating planes between each pair of satellites. A heuristic solution is introduced to choose these separating planes that lead to the convexification of the collision avoidance constraints. Additionally, convex constraints are imposed to guarantee that no collisions occur between discrete time samples. Campbell (2002) proposed a methodology utilizing Hamilton-Jacobian-Bellman optimality to generate a set of fuel- or time-optimal trajectories from an initial stable formation to a final stable formation in circular orbits. It is assumed that the fuel-optimal solution is an intuitive form of control input, in which, after each constant thrust input is applied, there is a period where the satellite “coasts” to the next switching point. The maneuver trajectories are calculated by optimizing the switch times using CW equations. But this methodology is limited to satellite formation in circular orbits. This methodology is then extended to satellite formations in elliptic orbits by Zanon and Campbell (2006). An example is presented of a tetrahedron formation in a highly elliptical reference orbit, with solutions to both formation minimum-time and minimum-fuel problems given. Cho et al. (2012) presented an analytic solution to the optimal reconfiguration problem of satellite formation flying in circular orbit in J_2 perturbation by using continuous and variable low-thrust propulsion.

Perturbations associated with differential gravity effects, the eccentricity effects of a chief orbit, and the J_2 effects were considered to obtain an approximated analytic solution to the optimal reconfiguration problem of formation flying using a power-limited low-thrust propulsion in Lee and Park (2011).

Linear relative dynamics models were used in the above linear optimization methods. However, these linearized relative motion equations ignore nonlinearity and orbital perturbations due to the Earth's oblateness or the J_2 effects. In order to obtain the accurate formation maneuver trajectories, some researchers investigated nonlinear trajectory optimization by using nonlinear dynamics.

Petit et al. (2001) used the Nonlinear Trajectory Generation (NTG) software package to generate constrained trajectories for satellite formation keeping and reconfiguration based on equations of satellite's absolute motion in inertia coordinate frame. The formulated optimal control problems were discretized into nonlinear programming problems. Sequential quadratic programming package NPSOL was used to solve nonlinear programming in NTG. Huntington et al. (2007b) proposed nonlinear fuel-optimal configuration method for tetrahedral formation based on Gauss variational equations. This configuration is to transfer a four-satellite formation from an initial parking orbit to a terminal reference orbit where the formation attains a desired tetrahedral shape at a specified point in the terminal reference orbit. The optimization problems were solved numerically by using Gauss pseudospectral method. The same method is used to solve the optimal formation reconfiguration problem in Huntington and Rao (2008). However, collision avoidance was not considered in both papers. Thevenet and Epenoy (2008) proposed a methodology to generate fuel-optimal maneuver trajectories for the deployment of satellite formation based on Gauss variational equations. The problem was addressed as an optimal control problem, the necessary optimality conditions of which were derived from Pontryagin's maximum principle. The resulting necessary optimality conditions were then solved with continuation-smoothing method. Mauro and Franco (2006) proposed a nonlinear low-thrust trajectory optimization method for formation flying using parallel multiple shooting. The problem was stated as the solution of an optimal control problem in which an objective function related to controls is minimized satisfying a series of constraints on the trajectory which are both differential and algebraic. The optimization problem was transcribed into a nonlinear programming problem with a parallel multiple shooting transcription method. The resulting nonlinear programming problems were then solved with an interior point method. However, the final relative state for each satellite was restricted to some prior fixed state, and the mass flow of the engine is not considered in this paper.

This chapter aims to design fuel-optimal low-thrust trajectories with high precision for formation maneuver by using the exact nonlinear relative dynamics in the presence of eccentricity and Earth's oblateness (Wu et al. 2009). In order to maneuver the satellites to a bounded desired formation, the energy-matching condition constraint and final geometry configuration constraints are included in the optimization problem (Wu et al. 2011). Furthermore, the collision avoidance constraints are considered to guarantee safe maneuvering among satellites.

The problem is numerically solved by a direct transcription method called Legendre pseudospectral method (Fahroo and Ross 2001; Elnagar et al. 1995; Qi et al. 2006).

Numerical methods for solving trajectory optimization problem can be grouped into two major categories: indirect methods and direct methods. A comprehensive survey of various numerical methods for solving trajectory optimization problems can be found in Betts (1998). In indirect methods, the optimality conditions are derived using calculus of variations. These necessary conditions lead to a nonlinear multipoint boundary-value problem. The main drawbacks of indirect method are that the radius of the convergent is usually very small and it is difficult to analytically derive the optimality condition for complex problem (Betts 1998). In direct methods, trajectory optimization problem is discretized into parameter optimization problem which results in nonlinear programming problem (NLP). NLP can be solved using appropriate optimization method such as those described in Betts (1998). Common discretization methods used in direct methods include trapezoidal, Hermite-Simpson, and Runge-Kutta. Direct methods are actively investigated by many researchers. There are two main reasons for the widespread use of direct methods. First, they can be applied without explicitly deriving the necessary optimality conditions. Second, direct methods do not require a prior specification of the arc sequence for problems with path inequalities.

In particular, a direct method that has shown tremendous promise is Legendre pseudospectral method. It has been applied to a wide variety of applications including atmospheric entry (Rao and Clarke 2002; Bollino et al. 2006) orbital transfer (Stanton and Proulx 2003; Ross et al. 2002), satellite formation design (Ross et al. 2002; Infeld et al. 2004), trajectory design for launch vehicle (Rea 2003; Ross et al. 2003), and satellite attitude control (Hui et al. 2007). In the Legendre pseudospectral method, the continuous state and control variables are approximated using a basis of globally orthogonal polynomials which results in improved accuracy and faster speed of convergence. The pseudospectral differentiation matrix then provides a numerical estimation of the derivatives at those points. The pseudospectral differentiation matrix transforms the differential equation of relative motion into a nonlinear programming problem.

4.2 Nonlinear Relative Motion Dynamics

In order to generate accurate formation maneuver trajectories, high-fidelity relative dynamics model is required. An exact nonlinear relative dynamics model developed by Xu and Wang (2008) is used in this chapter. In the presence of only J_2 perturbation, these developed dynamic equations are exact in describing the relative satellite motion in any eccentric orbits. This model is introduced in detail in Sect. 2.2. In this section, this model is modified to take into account of thruster dynamics.

The J_2 nonlinear ordinary differential equations describing relative satellite motion are as follows:

$$\begin{aligned}
\ddot{x}_j &= 2\dot{y}_j\omega_z - x_j(\eta_j^2 - \omega_z^2) + y_j\alpha_z - z_j\omega_x\omega_z - (\zeta_j - \zeta) \sin i \sin \psi - r(\eta_j^2 - \eta^2) + a_{j,x} \\
\ddot{y}_j &= -2\dot{x}_j\omega_z + 2\dot{z}_j\omega_x - x_j\alpha_z - y_j(\eta_j^2 - \omega_z^2 - \omega_x^2) + z_j\alpha_x - (\zeta_j - \zeta) \sin i \cos \psi + a_{j,y} \\
\ddot{z}_j &= -2\dot{y}_j\omega_x - x_j\omega_x\omega_z - y_j\alpha_x - z_j(\eta_j^2 - \omega_x^2) - (\zeta_j - \zeta) \cos i + a_{j,z}
\end{aligned} \tag{4.1}$$

where (x_j, y_j, z_j) denote the coordinates of the j th deputy satellite in LVLH frame and $(a_{j,x}, a_{j,y}, a_{j,z})$ are the control accelerations;

For the application under consideration here, the control accelerations are written as follows:

$$a_{j,x} = F_j u_{j,x} / m_j, a_{j,y} = F_j u_{j,y} / m_j, a_{j,z} = F_j u_{j,z} / m_j \tag{4.2}$$

where F_j is the magnitude of constant thrust, m_j is the satellite mass, and $(u_{j,x}, u_{j,y}, u_{j,z})$ are the components of the thrust direction.

During the burn phase of maneuver trajectory, the thrust magnitude is constant, but the thrust direction is steerable. Finally, the mass flow rate of the engine is described by the following equation:

$$\dot{m}_j(t) = -T_j / (g_0 I_{sp}) \tag{4.3}$$

where g_0 is the sea-level acceleration due to gravity and I_{sp} is the specific impulse of the engine.

4.3 Problem Formulation of Trajectory Optimization for Satellite Formation

Trajectory optimization is critical to stable initial satellite formation deployment and successful subsequent formation reconfiguration. The trajectory optimization problem for a fuel-optimal satellite formation maneuver is to determine the control input, the maneuver time, and the corresponding state trajectory so that the fuel consumption is minimized. The optimization problem is subject to constraints on nonlinear relative motion dynamics, given initial and final conditions, path constraints, and other constraints. The trajectory optimization problem for a fuel-optimal satellite formation maneuver can be formulated as follows.

Determine the thrust direction and the maneuver time that maximize the terminal mass of satellites, i.e., minimize the following objective function:

$$J = - \sum_{j=1}^{N_s} m_j(t_{j,f}) \tag{4.4}$$

The above-mentioned optimization problem is subject to constraints on relative motion dynamics, given initial and final conditions, and path and linking constraints. The details of others constraints are given as follows.

4.3.1 Initial Condition Constraints

The initial conditions for the trajectory optimization are the initial relative state and the initial mass of each deputy satellite. The initial conditions are given as follows:

$$x_j(t_0) = x_{j,0}, y_j(t_0) = y_{j,0}, z_j(t_0) = z_{j,0}, \dot{x}_j(t_0) = \dot{x}_{j,0}, \dot{y}_j(t_0) = \dot{y}_{j,0}, \dot{z}_j(t_0) = \dot{z}_{j,0} \quad (4.5)$$

$$m_j(t_0) = m_{j,0} \quad (4.6)$$

4.3.2 Final Condition Constraints

A few published works deal with the final conditions of trajectory optimization for satellite formation, but most of them set the final states of satellites as a priori fixed points. Richards et al. (2002) define many subsets of final states and perform the assessment for all subsets within the trajectory optimization process. Subsequently, the subset requiring the lowest overall fuel cost is selected. Huntington and Rao (2005) derive the final configuration constraints for tetrahedral formation. Two methods to deal with the final condition constraints are studied and compared in this chapter. In the first method, the final relative states are treated as variables in the optimization and are subjected to final geometry configuration constraints and energy-matching condition to obtain bounded desired final formation. In the second method, the final relative states are given as fixed points determined a priori by using an analytic (Sabol et al. 2001) or a numerical (Infeld et al. 2004) formation design method. The analytic formation design method proposed by Sabol et al. (2001) is used in this chapter. It can be seen from the simulation results presented later in this chapter that the first method can result in lower fuel consumption than the second method.

The energy-matching condition considering nonlinearity and orbit eccentricity for bounded relative satellite motion is derived by Gurfil (2005). This energy-matching constraint imposed at the end of maneuver for each satellite, shown in (4.7), is included in the optimization to obtain a bounded final formation.

$$\left| \frac{1}{2} \left\{ (\dot{x}_{j,f} - \omega_{z,f} y_{j,f} + \dot{r}_f)^2 + [\dot{y}_{j,f} + \omega_{z,f} (x_{j,f} + r_f)]^2 + \dot{z}_{j,f}^2 \right\} - \mu / \sqrt{(r_f + x_{j,f})^2 + y_{j,f}^2 + z_{j,f}^2} + \mu / 2a \right| \leq \varepsilon_1 \quad (4.7)$$

where ε_1 is a very small number which ensures that the optimization is feasible.

There are too many possibilities for formation geometry configuration, but circular formation and projected circular formation are the two most studied satellite formations (Sabot et al. 2001). The final geometry configuration constraints are derived for circular formation with three deputy satellites evenly spaced in a circle. These constraints can be easily extended to projected circular formation.

For circular formation (Sabot et al. 2001), one of the geometry constraints is as follows:

$$\left| \sqrt{x_j^2 + y_j^2 + z_j^2} - r_{cf} \right| \leq \varepsilon_2 \quad (4.8)$$

where r_{cf} is the radius of circular formation and ε_2 is a very small number which ensures that the optimization is feasible, because an exact circular formation does not exist when the chief satellite is in an elliptic orbit. Note that the above constraint imposed only on the final state is not sufficient to establish a circular formation. This constraint should also be imposed on several relative positions during one orbital period after the maneuver.

There are two planes on which the circular formation is possible. Both planes intersect the cross-track/along-track plane along the along-track axis, but one is inclined at a positive angle to that plane and the other is inclined at a negative angle. So another geometry constraint as shown in (4.9) should be included to obtain the final formation in a specified plane.

$$\begin{cases} x_{j,f} z_{j,f} > 0 & \text{for plane inclined at a positive angle} \\ \text{or } x_{j,f} z_{j,f} < 0 & \text{for plane inclined at a negative angle} \end{cases} \quad (4.9)$$

For the formation with three deputy satellites evenly spaced on the circle, the following constraints should also be included to ensure that the three satellites are evenly spaced in the circle.

$$\bar{d}_{12}(t_f) = \bar{d}_{13}(t_f) = \bar{d}_{23}(t_f) \quad (4.10)$$

where $\bar{d}_{p,q}$ denotes the distance between satellite p and satellites q .

4.3.3 Path Constraints

Firstly, the collision between the satellites should be avoided during the maneuver, so the following inequality constraints should be imposed on relative positions.

$$\bar{d}_{p,q}(t) \geq \bar{d}_{\text{safe}}(t), \quad \forall p, q \in [1, 2, \dots, N_s], p \neq q, \forall t \in [t_0, t_f] \quad (4.11)$$

Secondly, the mass of each satellite cannot fall below the dry mass. Defining the dry mass of the each satellite as $m_{j, \text{dry}}$, the following inequality constraints should be imposed on the mass of each satellite throughout the whole trajectory.

$$m_j(t) \geq m_{j,\text{dry}}, \quad \forall t \in [t_0, t_f] \quad (4.12)$$

Lastly, it is necessary to constrain the thrust direction vector to a constant unit length during the burn phase of the trajectory as follows:

$$u_{j,x}^2(t) + u_{j,y}^2(t) + u_{j,z}^2(t) = 1, \quad \forall t \in [t_0, t_f] \quad (4.13)$$

4.3.4 Linking Constraints

In order to reduce fuel consumption during maneuver, the thruster can fire several times such that the trajectory can be divided into several phases including coast phases and burn phases. During the coast phase, the thruster is turned off. While during the burn phase, the thrust is constant at its maximum value, but the thrust direction is steerable. So it is necessary to enforce linkage conditions at every phase boundary to ensure that the trajectory and mass of each satellite are continuous at the interface. These linkage conditions are enforced on the relative position, relative velocity, mass, and time and are given as follows:

$$\begin{aligned} x_j(t_f^v) &= x_j(t_0^{v+1}), y_j(t_f^v) = y_j(t_0^{v+1}), z_j(t_f^v) = z_j(t_0^{v+1}); \\ \dot{x}_j(t_f^v) &= \dot{x}_j(t_0^{v+1}), \dot{y}_j(t_f^v) = \dot{y}_j(t_0^{v+1}), \dot{z}_j(t_f^v) = \dot{z}_j(t_0^{v+1}); \\ m_j(t_f^v) &= m_j(t_0^{v+1}); \quad t_{j,f}^v = t_{j,0}^{v+1} \end{aligned} \quad (4.14)$$

The initial time and the terminal time of each phase are free, but the initial time of the first phase is fixed. In order to ensure that the time is increasing during the trajectory, the following inequality constraints are imposed on the time during each phase of the trajectory:

$$t_{j,f}^v > t_{j,0}^v \quad (4.15)$$

In summary, the trajectory optimization problem for satellite formation maneuver using continuous low thrust can be described as follows.

Problem Statement 4.1 The trajectory optimization problem for a fuel-optimal satellite formation maneuver is to determine the thrust direction and the maneuver time that maximize the terminal mass of satellites, i.e., minimize fuel consumption function as in (4.4). The optimization problem is subject to relative motion dynamics constraint (4.1)–(4.3); the initial relative state constraint (4.5) and the initial mass constraint (4.6); path constraints including collision avoidance constraint (4.11), mass constraints (4.12), and thrust direction constraint (4.13); final energy-matching constraint (4.7) and final formation geometry constraints (4.8)–(4.10); and phase linking constraints (4.14) and (4.15).

4.4 Introduction of Legendre Pseudospectral Method

As noted earlier, Legendre pseudospectral method has been widely used over the last few years to solve a variety of optimal control problems. A detailed description of the Legendre pseudospectral method for solving optimal control problems is provided in Fahroo and Ross (2001), Elnagar et al. (1995), and Qi et al. (2006). A brief review of this numerical method is given below.

The aforementioned single-phase trajectory optimization Problem Statement 4.1 can be written in the following general form. Determine the control direction, the maneuver time, and the corresponding state trajectory that minimize the following cost function:

$$J(\mathbf{x}, \mathbf{u}, t_f) = M[\mathbf{x}(t_f), t_f] \quad (4.16)$$

with $\mathbf{x} \in R^{N_s}$ and $\mathbf{u} \in R^{N_u}$ subject to the following dynamic constraint

$$\dot{\mathbf{x}}(t) = f[\mathbf{x}(t), \mathbf{u}(t)], \quad t \in [t_0, t_f] \quad (4.17)$$

and boundary conditions

$$\psi_{0l} \leq \psi_0[\mathbf{x}(t_0), t_0] \leq \psi_{0u} \quad (4.18)$$

$$\psi_{fl} \leq \psi_f[\mathbf{x}(t_f), t_f] \leq \psi_{fu} \quad (4.19)$$

with $\psi_0 \in R^p$ with $p \leq n$ and $\psi_f \in R^q$ with $q \leq n$ and mixed state-control path constraint

$$h_l \leq h[\mathbf{x}(t), \mathbf{u}(t), t] \leq h_u \quad (4.20)$$

It is assumed that the above functions are continuously differentiable with respect to their arguments.

The above trajectory optimization problem can be discretized into nonlinear programming by the Legendre pseudospectral method. In the Legendre pseudospectral method, the Legendre–Gauss–Lobatto (LGL) points are used. These points $\tau_l, l = 0, \dots, N$, which are distributed on the interval $[-1, 1]$, are defined as follows:

$$\tau_0 = -1, \quad \tau_N = 1 \quad (4.21)$$

and for $1 \leq l \leq N-1$, τ_l are the zeros of $\dot{L}_N$, the derivative of the Legendre polynomial L_N . The discretization process begins by approximating the continuous state and control variables by N th polynomials of the form.

$$\mathbf{x}(\tau) \approx \mathbf{x}^N(\tau) = \sum_{l=0}^N \mathbf{x}(\tau_l) \phi_l(\tau) \quad (4.22)$$

$$\mathbf{u}(\tau) \approx \mathbf{u}^N(\tau) = \sum_{l=0}^N \mathbf{u}(\tau_l) \phi_l(\tau) \quad (4.23)$$

where for $l = 0, 1, \dots, N$

$$\phi_l(\tau) = \frac{1}{N(N+1)L_N(\tau_l)} \frac{(\tau^2 - 1)\dot{L}_N(\tau)}{\tau - \tau_l} \quad (4.24)$$

are the Lagrange interpolating polynomials of order N .

The dynamic equations are discretized by imposing the condition that the derivatives of the state approximations satisfy the differential equations exactly at the node points. Thus, the derivative of $\mathbf{x}^N(\tau)$ in terms of $\mathbf{x}(\tau)$ at the collocation point τ_k can be obtained by differentiating (4.22) and evaluating the result at τ_k . The result is a matrix multiplication given by

$$\dot{\mathbf{x}}(\tau_k) \approx \dot{\mathbf{x}}^N(\tau_k) = \sum_{l=0}^N \mathbf{x}(\tau_l) \dot{\phi}_l(\tau_k) = \sum_{l=0}^N D_{kl} \mathbf{x}(\tau_l) \quad (4.25)$$

For LGL points, $D_{kl} = \dot{\phi}_l(\tau_k)$ are the entries of the $(N+1) \times (N+1)$ differentiation matrix $\mathbf{D}$.

$$\mathbf{D} := [D_{kl}] := \begin{cases} \frac{L_N(\tau_k)}{L_N(\tau_l)} \frac{1}{\tau_k - \tau_l} & k \neq l \\ -\frac{N(N+1)}{4} & k = l = 0 \\ \frac{N(N+1)}{4} & k = l = N \\ 0 & \text{otherwise} \end{cases} \quad (4.26)$$

Because the trajectory optimization problem presented in (4.16)–(4.20) is formulated over the time interval $[t_0, t_f]$, the LGL points lie in the interval $[-1, 1]$; the following transformation is used to express the problem for $\tau \in [\tau_0, \tau_N] = [-1, 1]$:

$$t = \frac{[(t_f - t_0)\tau + (t_f + t_0)]}{2} \quad (4.27)$$

The mixed state-control path constraints and the boundary conditions can also be discretized by evaluating these inequalities at the LGL nodes. Thus, the single-phase trajectory optimization problem can now be discretized as follows:

Find the $(N+1)(N_x + N_u) + 1$ vector $\mathbf{X}_{\text{opt}}$

$$\mathbf{X}_{\text{opt}} = [\mathbf{x}(\tau_0), \mathbf{x}(\tau_1), \dots, \mathbf{x}(\tau_N); \mathbf{u}(\tau_0), \mathbf{u}(\tau_1), \dots, \mathbf{u}(\tau_N); t_f] \quad (4.28)$$

which minimizes

$$J(\mathbf{X}_{\text{opt}}) = M(\mathbf{x}(\tau_N), t_f) \quad (4.29)$$

subject to

$$\sum_{l=0}^N D_{kl} \mathbf{x}(\tau_l) - \frac{t_f - t_0}{2} f(\mathbf{x}(\tau_k), \mathbf{u}(\tau_k), \tau_k) = 0 \quad (4.30)$$

$$\psi_{0l} \leq \psi_0(\mathbf{x}(\tau_0), \tau_0) \leq \psi_{0u} \quad (4.31)$$

$$\psi_{fl} \leq \psi_f(\mathbf{x}(\tau_N), \tau_N) \leq \psi_{fu} \quad (4.32)$$

$$h_l \leq h_k(\mathbf{x}(\tau_k), \mathbf{u}(\tau_k)) \leq h_u \quad k = 0, \dots, N. \quad (4.33)$$

Remark 4.1 The Legendre pseudospectral method can be extended to problems with multiple phases in a straightforward manner (Ross et al. 2003). The pseudospectral implementation of trajectory optimization with multiple phases is treated by defining each phase separately and then linking these phases with a set of phase boundary conditions.

4.5 Computational Considerations of Nonlinear Programming Problem

This section discusses issues related to the implementation and solution of the optimization problem presented in the preceding sections. The methods presented here do not change the basic constraints already described, but assist greatly in obtaining solutions within a practical and reasonable computational time interval.

4.6 Scaling of Nonlinear Programming Problem

Scaling a nonlinear programming problem is very important in order to obtain robust and rapid convergence to the final solution. Betts (2001) commented that “Scaling affects everything! Poor scaling can make a good algorithm bad. Scaling changes the convergence rate, termination tests, and numerical conditioning.” One way to construct a well-scaled problem is to normalize the independent variables to have the same range (Betts 2001), e.g., $0 \leq \tilde{x}_k \leq 1$, where $\tilde{x}_k$ denotes the scaled variables. So the following normalizations are used in this work.

$$\tilde{x} = x/\bar{d}_{\max}, \tilde{y} = y/\bar{d}_{\max}, \tilde{z} = z/\bar{d}_{\max} \quad (4.34)$$

$$\tilde{m} = m/m_0, \quad \tilde{t} = t/T_c \quad (4.35)$$

The magnitude of the maximum relative velocities during a maneuver is in the order of 1, so it is not necessary to normalize the relative velocities.

4.6.1 Initial Guess

The Legendre pseudospectral method has a larger radius of convergence than other numerical methods, and it may not require a good initial guess for convergence. However, an educated initial guess does improve convergence rate and robustness.

In this section, an initial guess is generated by integrating the relative dynamics numerically using an arbitrary control direction. The generated states and control direction thus satisfy the initial conditions and the differential dynamic constraints, but do not satisfy the terminal constraints and are generally non-optimal. Using this initial guess, TOMLAB/SNOPT (Holmström et al. 2008) is able to solve the resulting NLP problems within several minutes. A better initial guess could be the optimized trajectory solution obtained for a relative dynamics model such as CW equations. The trajectory generated by the presented algorithm without considering collision avoidance constraint would also provide good initial guess for the trajectory optimization problem with collision avoidance constraint.

4.6.2 Implementation

The optimization was carried out with the TOMLABTM version of the NLP solver SNOPT using default optimality and feasibility tolerances. TOMLAB/SNOPT is a software package for solving large-scale optimization problems (linear and nonlinear programs). It is especially effective for nonlinear problems whose functions and gradients are expensive to evaluate. So TOMLAB/SNOPT is well suited to solve the proposed trajectory optimization problem after it is transcribed into a large-scale nonlinear programming problem.

4.7 Illustrative Examples

In this section, examples are presented for a wide class of satellite formation reconfiguration problems. Firstly, formation reconfigurations involving two satellites (one chief satellite and one deputy satellite) are studied. As mentioned earlier,

in order to conserve fuel during the maneuver, the thruster can be fired several times during the trajectory according to certain phase sequence. Trajectory optimizations with the following different phase sequences are studied and compared.

1. One phase: burn;
2. Two phases: coast–burn;
3. Three phases: burn–coast–burn; and
4. Four phases: coast–burn–coast–burn.

Secondly, formation reconfigurations involving multiple satellites, such as four satellites, are studied. Finally, the collision avoidance capability of the proposed method is also demonstrated.

The chief satellite orbit is characterized by the following orbit elements:

$$a = 7100 \text{ km}, e = 0.001, i = 45^\circ, \varpi = 30^\circ, \Omega = 45^\circ, \theta = 0^\circ \quad (4.36)$$

All the deputy satellites have been considered with the same following parameters:

$$m_0 = 50 \text{ kg}, T = 50 \text{ mN}, I_{\text{sp}} = 1000 \text{ s} \quad (4.37)$$

For easy of presentation, the parameters of deputy satellites, such as the masses, maximum thrusts, and thruster's specific impulse, are assumed to be identical. In practice, there will be variation. The proposed trajectory optimization method also applies to the case with different parameters. The above control force can be realized with an actual electric propulsion system (Polzin 2011; Pillet et al. 2006; Pencil et al. 2004).

4.7.1 Example 1: Scenario of Two Satellites, One Burn Phase

In this example, the deputy satellite in a circular formation of 1000 m radius is maneuvered to another circular formation with radius of 2000 m. During this maneuver with one burn phase, the thrust is constant at its maximum value. The number of LGL points is chosen as 64. More accurate results can be obtained in this NLP problem with more LGL points at the expense of computational cost. The clock frequency of the CPU used is 1.86 GHz. The optimization can be completed within 30 s even without using analytic first-order derivatives for both the constraints Jacobian and the gradient of the objective function. The initial relative state of deputy satellite with respect to chief satellite is given as follows:

$$x_0 = 500 \text{ m}, \dot{x}_0 = 0 \text{ m/s}, y_0 = 0 \text{ m}, \dot{y}_0 = 1 \text{ m/s}, z_0 = 866.0254 \text{ m}, \dot{z}_0 = 0 \text{ m/s} \quad (4.38)$$

As noted earlier, two different methods of handling the final condition constraints are considered. In the first method, the final relative state is treated as a variable in the optimization problem and subjected to final geometry configuration constraints and energy-matching condition to obtain bounded desired final formation. In the second method, the final relative state is determined by an analytic formation design method (Sabot et al. 2001). The computed final relative state for the second method is given as follows:

$$x_f = 1000 \text{ m}, \dot{x}_f = 0 \text{ m/s}, y_f = 0 \text{ m}, \dot{y}_f = 2 \text{ m/s}, z_f = 1732.1 \text{ m}, \dot{z}_f = 0 \text{ m/s} \quad (4.39)$$

The fuel-optimal trajectories generated for the two cases are shown in Figs. 4.1 and 4.2, respectively. Maneuver time and fuel consumption are shown in Table 4.1. For ease of interpretation, ΔV is calculated approximately by assuming constant mass of satellite during the formation maneuver. As observed from the result, the first method results in 48.35 % fuel saving as compared to the second method. So it can be concluded that for formation maneuver with a single burn phase, the first method, with substantially lower fuel consumption, is much better than the second method. The components of the control direction for the first method are shown in Fig. 4.3.

An open-loop guidance demonstration is performed using the control direction generated by the Legendre pseudospectral method. The Legendre pseudospectral method gives a state estimation and control only at the LGL points. Cubic spline

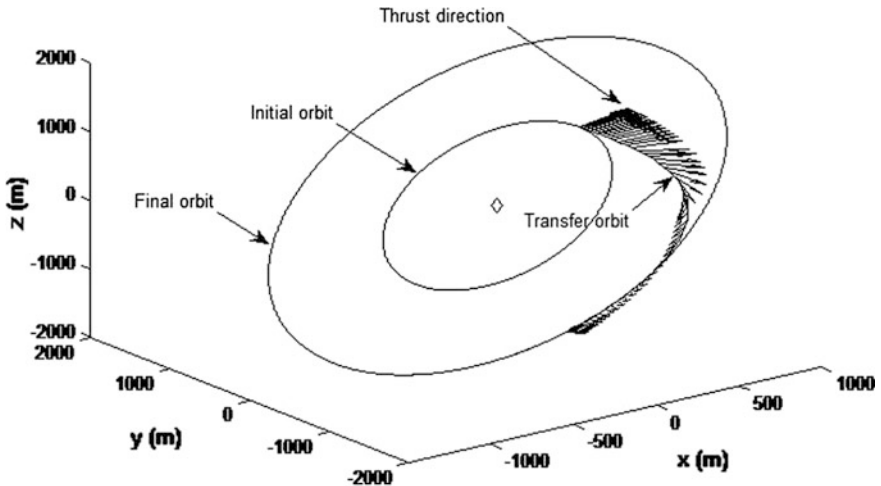


Fig. 4.1 Trajectories of formation reconfiguration with free relative final state

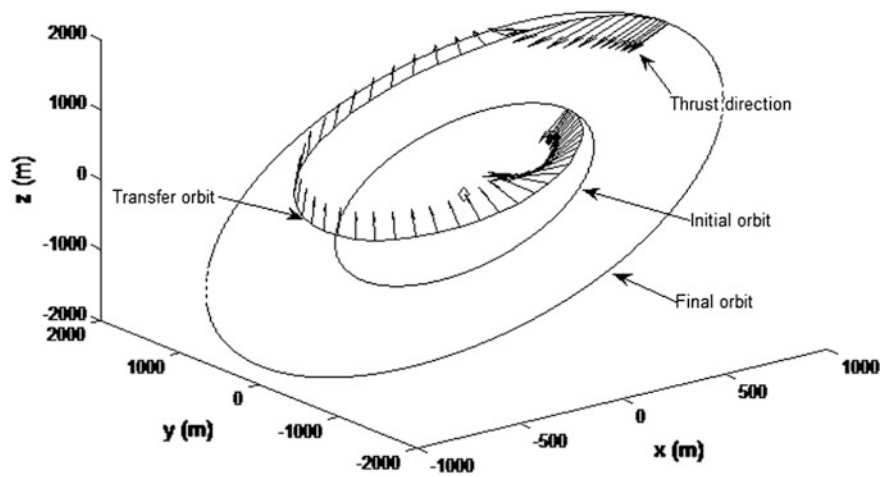


Fig. 4.2 Trajectories of formation reconfiguration with given final relative states prior

Table 4.1 Fuel consumption and maneuver time for single-phase maneuver

	Fuel consumption		Maneuver time, s
	Δm , kg	ΔV , m/s	
Free final state	0.01131	2.217	2221
Fixed final state	0.02190	4.292	4283

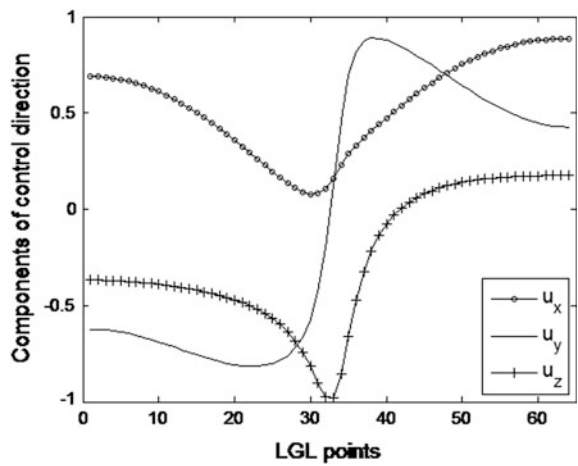
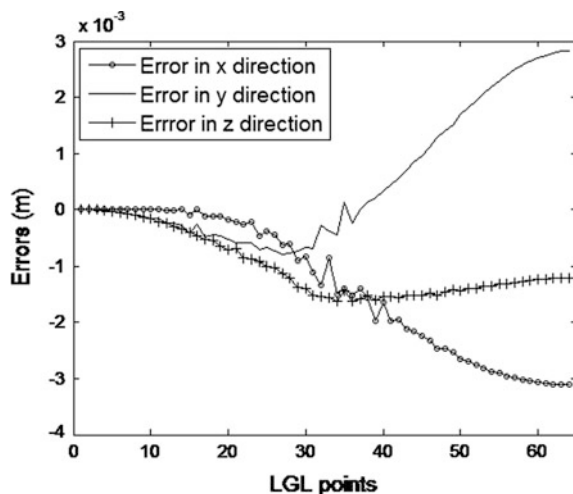


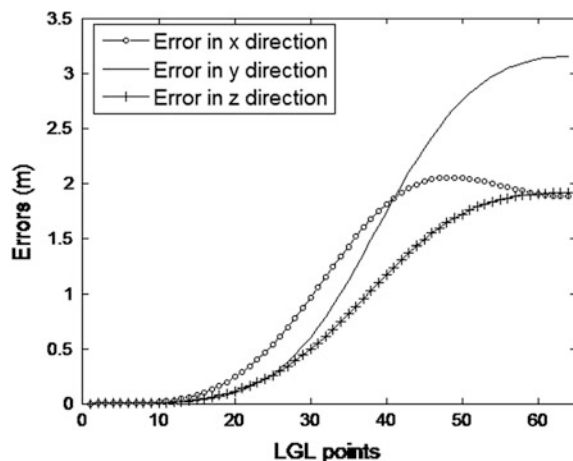
Fig. 4.3 The control direction of thrust

Fig. 4.4 Error history with dynamics considering J_2 perturbation



interpolation is used to estimate the controls between the LGL points. In the simulation, the optimized control force is executed on the deputy satellite in an open-loop way, with no in-flight corrections, and the actual relative trajectory is generated by integrating the absolute equation of motion including J_2 effect for each satellite using MATLAB command “ode45”. The errors between the actual trajectory and the optimized trajectory are plotted in Fig. 4.4. For comparison, another trajectory is generated by the Legendre pseudospectral method using nonlinear relative dynamics without J_2 effect. The errors between the actual trajectory and this optimized trajectory are shown in Fig. 4.5. As can be observed from the results, the errors with relative dynamics considering J_2 effect are much smaller than that without considering J_2 effect. It can be concluded that nonlinear relative satellite

Fig. 4.5 Error history with dynamics not considering J_2 perturbation



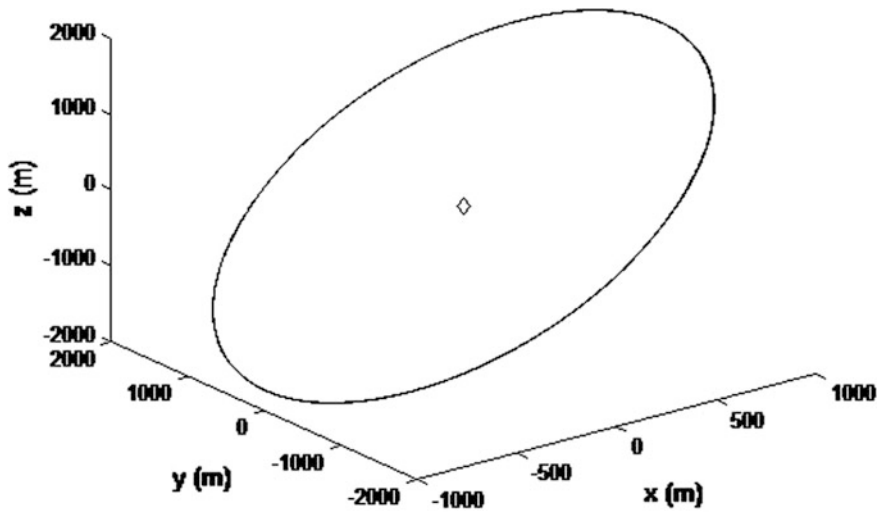


Fig. 4.6 Trajectories of final formation propagated for 1 day

dynamics model considering J_2 effect is necessary to obtain accurate trajectory for satellite formation maneuver.

In order to validate the effectiveness of the energy-matching condition in obtaining bounded final formation, the generated final formation is propagated in the dynamics environment without considering J_2 effect for 24 h. The propagated formation is plotted in Fig. 4.6. As can be seen from the result, the drift of the formation is very small, and a bounded formation is obtained. As noted earlier, the exact energy-matching condition considering J_2 effect is presently under development.

Another kind of pseudospectral method developed for solving trajectory optimization problem is Gauss pseudospectral method (GPM) (Benson et al. 2006), which is similar to the Legendre pseudospectral method (LPM). In order to further validate the efficiency of pseudospectral method in designing fuel-optimal maneuvers for satellite formation reconfiguration problem, GPM is also utilized to solve the above formation reconfiguration problem with equivalent initial guess as for LPM. Relative position error histories for LPM and GPM are generated using the open-loop guidance method as in the above example. Maximum relative position errors and computational times are shown in Table 4.2. It is shown Table 4.2 that LPM and GPM are comparable in computational efficiency and trajectory accuracy. Note that it is shown in Huntington et al. (2007a) that GPM can produce significantly more accurate costate estimations than LPM without incurring

Table 4.2 Computation times and maximum position errors for LPM and GPM

	CPU times, s		Maximum position error, m	
	LPM	GPM	LPM	GPM
$N = 15$	3.06	4.08	2.802×10^{-1}	3.420×10^{-1}
$N = 30$	10.21	8.06	1.65×10^{-2}	1.54×10^{-2}
$N = 45$	17.20	16.06	2.7×10^{-3}	3.0×10^{-3}

additional computational burden. For the details of performance comparison between LPM and GPM, please refer to Huntington et al. (2007a).

4.7.2 Example 2: Scenario of Two Satellites, Two Phases: Coast–Burn

In this example, the formation maneuver mission, the initial condition, and the final condition are the same as those in the previous example. The only difference is the phase sequence. Before the burn phase, one coast phase is added to the trajectory so that the satellite can choose a right time to leave the original formation so as to save fuel. The number of LGL points for each phase is 64. Fuel-optimal trajectories generated by using the two different methods dealing with the final condition constraints are plotted in Figs. 4.7 and 4.8, respectively. Coast time, burn time, and fuel consumption are shown in Table 4.3. As seen from simulation results, the maneuver with two phases can save about half the fuel required by that with a single burn phase.

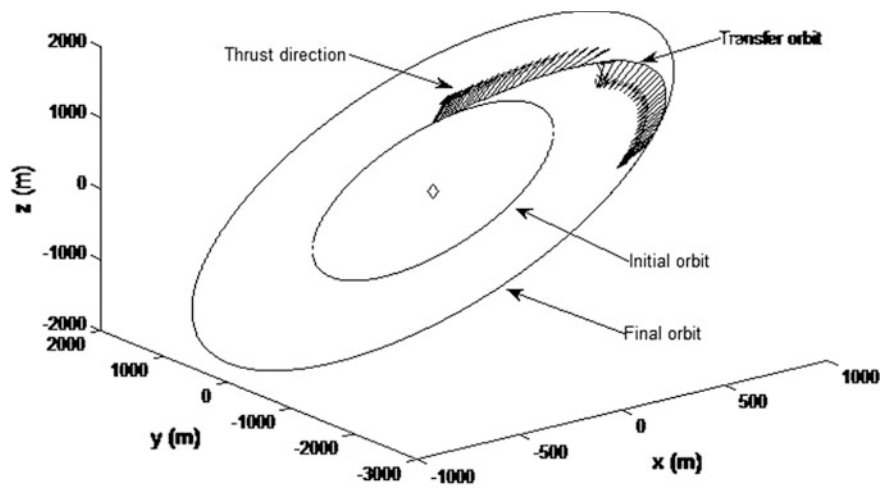


Fig. 4.7 Trajectories of formation reconfiguration with free relative final state

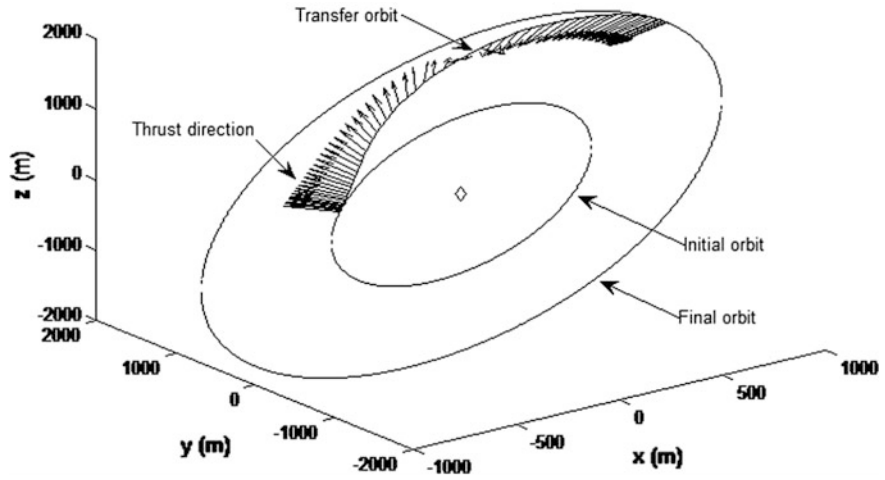


Fig. 4.8 Trajectories of formation reconfiguration with given relative final state

Table 4.3 Fuel consumption and maneuver time for two-phase maneuver

Cases	Fuel consumption		Maneuver time, s	
	Δm , kg	ΔV , m/s	Coast	Burn
Free final state	0.01046	2.050	5255	2050
Fixed final state	0.01069	2.095	4149	2095

4.7.3 Example 3: Scenario of Two Satellites, Three Phases: Burn–Coast–Burn

In this example, the trajectory is divided into three phases: burn–coast–burn. Fuel-optimal trajectories generated by using the two different methods dealing with final condition constraints are shown in Figs. 4.9 and 4.10, respectively. Maneuver time and fuel consumption are shown in Table 4.4. As observed from the result, the first method uses 0.4 % less fuel than the second method, and the maneuver with three phases requires less fuel than maneuver with two phases.

4.7.4 Example 4: Scenario of Two Satellites, Four Phases: Coast–Burn–Coast–Burn

In this example, the trajectory is divided into four phases: coast–burn–coast–burn. Fuel-optimal trajectories generated by using the two different methods dealing with final condition constraints are plotted in Figs. 4.11 and 4.12, respectively. Coast times, burn times, and the fuel consumption in each case are shown in Table 4.5.

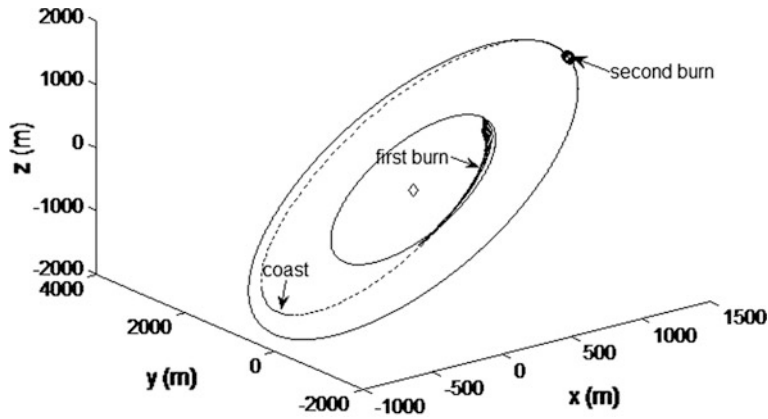


Fig. 4.9 Trajectories of formation reconfiguration with free relative final state

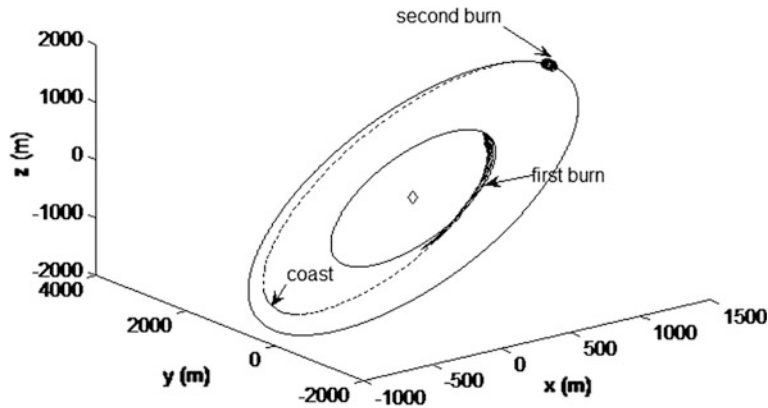


Fig. 4.10 Trajectories of formation reconfiguration with given relative final state

Table 4.4 Fuel consumption and maneuver time for three-phase maneuver

Cases	Fuel Consumption		Maneuver time, s			
	Δm , kg	ΔV , m/s	1st burn	Coast	2nd burn	Total
Free final state	0.006671	1.308	1243	4692	65	6000
Fixed final state	0.006698	1.313	1248	4429	64	5741

It is observed from the result that the first method can save 3.4 % fuel as compared to the second method, and a maneuver with four phases requires less fuel than a maneuver with three phases.

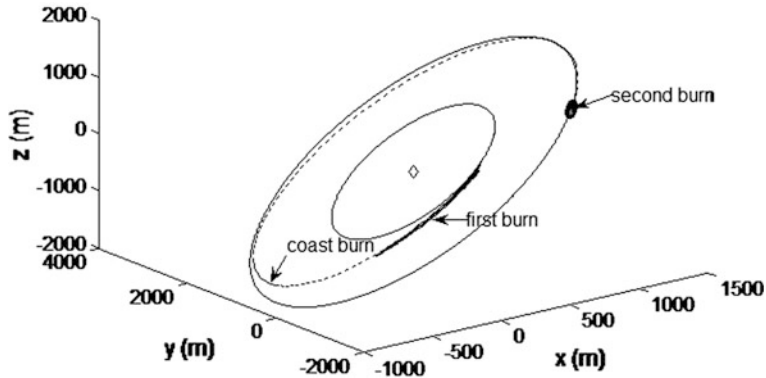


Fig. 4.11 Trajectories of formation reconfiguration with free relative final state

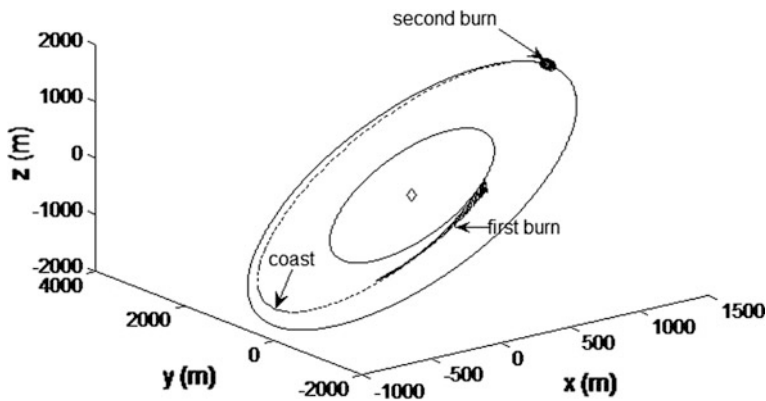


Fig. 4.12 Trajectories of formation reconfiguration with given relative final state

Table 4.5 Fuel consumption and maneuver time for four-phase maneuver

Cases	Fuel consumption		Maneuver time, s			
	Δm (kg)	ΔV (m/s)	1st coast	1st burn	2nd coast	2nd burn
Free final state	0.005181	1.016	1146	946	4198	73
Fixed final state	0.005357	1.050	1053	988	4146	65

In this chapter, several structures of the trajectories up to coast–burn–coast–burn with progressive but diminishing improvement on fuel consumption have been studied. The fuel consumption of the trajectory of coast–burn–coast–burn sequence

is close to minimum. As observed in Huntington and Rao (2005, 2008), although the satellite formation maneuver trajectories are divided into six phases and nine phases, there are only one or two actual burn phases in the optimized trajectories, and the optimal durations of other left burn phases are zero.

4.7.5 Example 5: Scenario of Formation Reconfiguration Involving Four Satellites

This example is to validate the applicability of the proposed trajectory optimization method for multi-satellite formation. Formation reconfiguration is studied for formation with four satellites (one chief satellite and three deputy satellites). The three deputy satellites are maneuvered from a circular formation with radius of 1000 m to another circular formation with radius of 2000 m, and the three deputy satellites are evenly spaced in the final circular formation. The initial relative state of each deputy satellite is as follows:

$$\begin{aligned} x_{1,0} &= 500, \dot{x}_{1,0} = 0, y_{1,0} = 0, \dot{y}_{1,0} = 1, z_{1,0} = 866, \dot{z}_{1,0} = 0 \\ x_{2,0} &= -250, \dot{x}_{2,0} = 0.433, y_{2,0} = 866, \dot{y}_{2,0} = 0.5, z_{2,0} = 433, \dot{z}_{2,0} = 0.75 \\ x_{3,0} &= -250, \dot{x}_{3,0} = 0.433, y_{3,0} = -866, \dot{y}_{3,0} = 0.5, z_{3,0} = 433, \dot{z}_{3,0} = -0.75 \end{aligned} \quad (4.40)$$

where the unit of relative positions is m and that of relative velocities is m/s.

As can be concluded from the earlier examples, for a maneuver with more than one phase, the optimization with a free final state does not result in significant fuel saving as compared to that with fixed final state. Thus, the final state for a maneuver with multiple phases is chosen to be fixed. In this example, the final relative states satisfying the desired final formation are given in (4.41) by applying the analytic formation design method (Sabot et al. 2001), and the trajectory is divided into two phases (coast–burn).

$$\begin{aligned} x_{1,f} &= -110.8, \dot{x}_{1,f} = -0.9942, y_{1,f} = -1988.7, \dot{y}_{1,f} = 0.2216, z_{1,f} = -191.4, \dot{z}_{1,f} = -1.7195; \\ x_{2,f} &= 914.4, \dot{x}_{2,f} = 0.4051, y_{2,f} = 800.6, \dot{y}_{2,f} = -1.8289, z_{2,f} = 1587.1, \dot{z}_{2,f} = 0.6940; \\ x_{3,f} &= -806.9, \dot{x}_{3,f} = 0.5918, y_{3,f} = 1183.3, \dot{y}_{3,f} = 1.6138, z_{3,f} = -1394.9, \dot{z}_{3,f} = 1.0231 \end{aligned} \quad (4.41)$$

where the unit of relative positions is m and that of relative velocities is m/s.

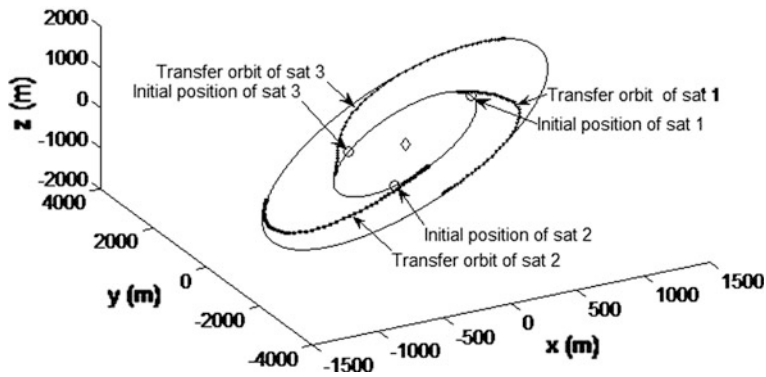


Fig. 4.13 Trajectories of formation reconfiguration of four satellites

Table 4.6 Fuel consumption and maneuver time for four-satellite formation (the units of Δm and ΔV are kg and m/s, respectively)

Deputy satellite 1				Deputy satellite 2				Deputy satellite 3			
Fuel		Time, s		Fuel		Time, s		Fuel		Time, s	
Δm	ΔV	Coast	Burn	Δm	ΔV	Coast	Burn	Δm	ΔV	Coast	Burn
0.0106	2.08	5918	2081	0.0109	2.13	5821	2131	0.0108	2.11	5804	2110

Fuel-optimal trajectories generated for three deputy satellites are shown in Fig. 4.13. The corresponding maneuver times and fuel consumptions are shown in Table 4.6.

4.7.6 Example 6: Scenario of Collision Avoidance Validation

The final example demonstrates the effectiveness of the proposed method in meeting collision avoidance requirement. The three deputy satellites are maneuvered from a circular formation with a radius of 500 m to another circular formation with a radius of 1000 m using only one burn phase. The initial and final positions as well as velocities are chosen a priori.

The optimized trajectories of each satellite obtained without considering the collision avoidance are shown in Fig. 4.14, and the distances between the deputy satellites during the maneuver are plotted in Fig. 4.15. As observed from the simulation results, the smallest distance between the 1st and the 3rd satellite is only 43 m. This separation is hazardous, and so collision avoidance constraint should be

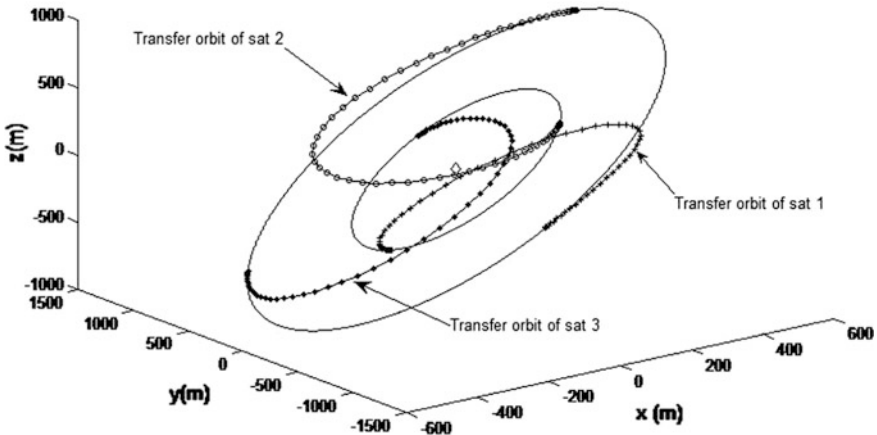


Fig. 4.14 Trajectories of formation maneuver without collision avoidance constraints

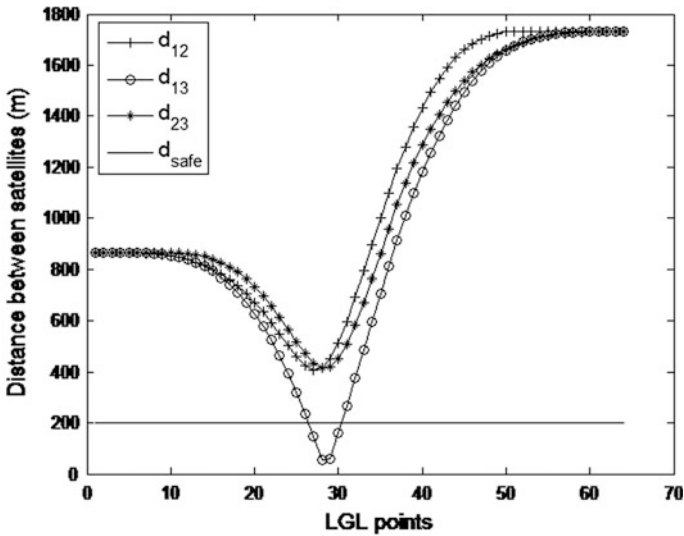


Fig. 4.15 Satellite relative distances during maneuver without collision avoidance constraints

included in the optimization. In the ensuing optimization problem the safe distance is assumed to be $d_{safe} = 200$ m. The optimized trajectories obtained by incorporating collision avoidance requirement are shown in Fig. 4.16, and the corresponding distances between deputy satellites during the maneuver are shown in

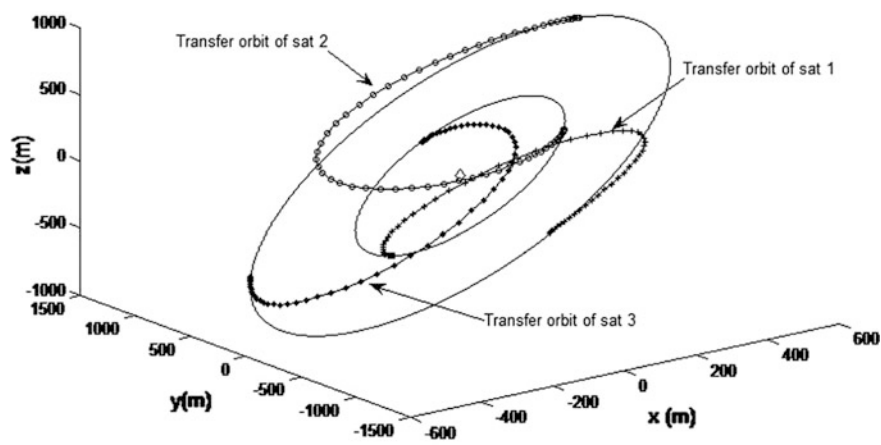


Fig. 4.16 Trajectories of formation with collision avoidance constraints

Fig. 4.17 Satellite relative distances during maneuver with collision avoidance constraints

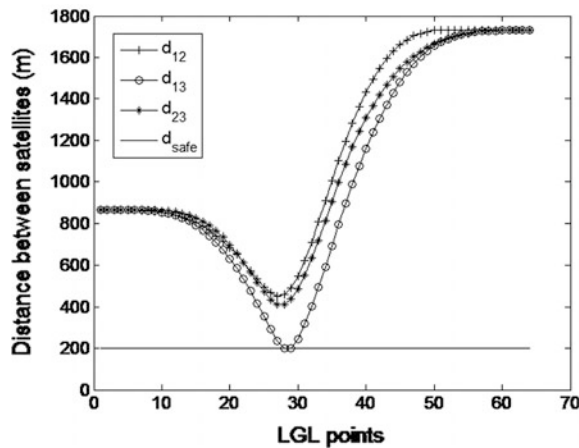


Fig. 4.17. As seen from the results, the distances between satellites are always greater than or equal to the required safe distance. It can be concluded that the proposed method can perform safe formation maneuver. As observed from Table 4.7, the fuel consumptions for the two cases are almost the same, but the method that incorporates the collision avoidance constraints can guarantee collision avoidance during the maneuver.

Table 4.7 Fuel consumption of each satellite during maneuver

	Fuel consumption of satellite 1		Fuel consumption of satellite 2		Fuel consumption of satellite 3	
	Δm , kg	ΔV , m/s	Δm , kg	ΔV , m/s	Δm , kg	ΔV , m/s
First maneuver	0.0255	4.998	0.0256	5.018	0.0255	4.998
Second maneuver	0.0258	5.057	0.0278	5.449	0.0267	5.233

4.8 Conclusions

The use of Legendre pseudospectral method to design fuel-optimal trajectories for satellite formation maneuver subject to collision avoidance is described and demonstrated in this chapter. Continuous low-thrust propulsion is used in order to obtain high maneuver precision. The trajectory optimization problem is reduced into nonlinear programming problem which is solved numerically by TOMLAB/SNOPT. High precision maneuver is obtained by using a newly developed exact relative dynamics considering eccentricity, nonlinearity, and Earth’s oblateness. Bounded desired final formation is acquired by including the energy-matching condition and final geometry configuration constraints. Extensive simulation results demonstrate the applicability of Legendre pseudospectral method to optimal formation trajectory design.

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Chapter 5

Optimal Control for Satellite Formation Keeping

This chapter presents two kinds of optimal control-based satellite formation keeping strategies. Firstly, a real-time optimal control law in a leader–follower frame is developed for satellite formation keeping in elliptic orbits. The proposed control law can prioritize between fuel consumption and control accuracy subject to the control saturation constraints. Then, decentralized control using only local relative measurements is proposed for satellite formation keeping in mildly elliptic orbits. Optimal guaranteed cost control law is applied to the resulting linear time-varying (LTV) system with uncertain parameters. The proposed decentralized formation control architecture is completely decentralized and scalable.

5.1 Introduction

Formation will drift apart slowly due to various environmental perturbations after the desired formation is established. Thus, satellite formation keeping strategy is required. Various continuous control approaches (De Queiroz et al. 2000; Vignal and Pernicka 2006; Gurfil et al. 2002; Mesbahi and Hadaegh 2001; Yeh et al. 2002; Massey and Shtessel 2005; Subbarao and Welsh 2008; Beard et al. 2001; Vassar and Sherwood 1985; Redding et al. 1989; Kapila et al. 2000; Capó-Lugo and Bainum 2007; Liu and Kumar 2012; Wang et al. 2012; Carpenter 2002; Belanger et al. 2006; Lavaei et al. 2008; Chung et al. 2009; Ren and Beard 2004; Ren 2007; Gurfil and Mishne 2007; Smith and Hadaegh 2005; Min et al. 2011; Ramirez-Riberos et al. 2010; Shan 2008; Breger and How 2007) have been proposed to investigate the problem of formation keeping. A survey of various control design methodologies for satellite formation using non-impulsive propulsion system can be found in Scharf et al. (2004). These approaches can be roughly divided into two categories: leader–follower control and decentralized control. In this chapter, firstly a real-time fuel-optimal formation keeping approach in the leader–follower frame is designed using low-thrust propulsion system. Then, a

decentralized continuous formation keeping approach using local relative measurements is proposed based on algebraic graph theories. Before the two control approaches are presented, the literature review of leader–follower control and decentralized control for satellite formation keeping is firstly given below.

5.1.1 Leader–Follower Formation Keeping Approaches

Most of control strategies for satellite formation keeping use leader–follower architecture in which one satellite is designated as leader, and the remaining satellites are manipulated to follow the same leader. Numerous contributions (De Queiroz et al. 2000; Vignal and Pernicka 2006; Gurfil et al. 2002; Mesbahi and Hadaegh 2001; Yeh et al. 2002; Massey and Shtessel 2005; Subbarao and Welsh 2008; Shan 2008; Beard et al. 2001; Vassar and Sherwood 1985; Redding et al. 1989; Kapila et al. 2000; Capó-Lugo and Bainum 2007; Breger and How 2007; Liu and Kumar 2012; Wang et al. 2012) to the leader–follower literature differ primarily in the types of formation tracking control laws designed for the followers. In this regard, most of the modern control paradigms have been investigated to solve this problem.

The problem of relative position control for satellite formation flying was considered in De Queiroz et al. (2000). Specifically, the full nonlinear dynamics describing the relative positioning of satellite formation flying are used to develop a Lyapunov-based, nonlinear adaptive control law that guarantees global asymptotic convergence of the position tracking error in the presence of unknown constant, or slow-varying satellite masses, disturbance forces, and gravity forces. Vignal and Pernicka (2006) compared two active control system designs that can be used to maintain formation composed of microsatellites with limited on–off thrusting capabilities in low Earth orbit. The two evaluated designs are based on a linear approach using optimal control theory and on a nonlinear approach based on Lyapunov stability concepts. The final selection is a trade-off between fuel consumption and controller robustness based on model uncertainties. Gurfil et al. (2002) presented a nonlinear adaptive neural control methodology for spacecraft formation flying in the spatial elliptic restricted three-body problem. This controller consists of an approximate dynamic model inversion, linear compensation of the ideal feedback linearized model, and an adaptive neural network-based element designed to compensate for the model inversion errors. Mesbahi and Hadaegh (2001) also presented several results in the leader–follower framework for satellite formation flying control. Ideas from elementary graph theory and linear matrix inequalities are combined with logic-based switching to shed light on the class of control designs that are feasible through leader–follower for a representative set of formation flying scenarios. A tracking control design using sliding mode techniques was derived to control a desired satellite formation in Yeh et al. (2002). CW equations are used to model the follower satellite motion relative to the leader. The control law is modified to account for discontinuous nature of the control forces

available with the satellite propulsive thrusters. This study is further improved in Massey and Shtessel (2005) via application of recently developed continuous sliding mode control algorithms modulated by pulse-width modulation (PWM), allowing high tracking accuracy in the presence of bounded matched disturbances and uncertainties and providing a required duty cycle in a control command to the jet thrusters.

Most of the above-cited formation keeping approaches dealt with model uncertainties and external disturbances. However, fuel consumption is a major concern for satellite formation keeping mission. Linear quadratic regulator (LQR) is an optimal control approach, which saves fuel consumption by increasing the relative weight matrix of control input over state error in the cost function, and it was studied by several researchers for formation keeping (Vassar and Sherwood 1985; Redding et al. 1989; Kapila et al. 2000; Capó-Lugo and Bainum 2007). But solving the LQR problem for LTV system, modeling the relative motion in elliptic orbits requires an efficient computation of the Riccati equations (Capó-Lugo and Bainum 2007) and solving Riccati equations in real time is computation-intensive. Furthermore, LQR cannot handle control input saturation efficiently, which is very important for low-thrust formation keeping system. To overcome the mentioned limitations of LQR for LTV system, Breger and How (2007) proposed a real-time optimal control for satellite formation keeping in elliptic orbits. However, zero-order-hold discretized model is required for the optimization-based controller in Breger and How (2007). Hence, large discretization error will result when the discretization step is large.

5.1.2 *Decentralized Formation Keeping Approaches*

Although the leader–follower approach is easy to understand and implement, there are limitations. The leader–follower approach is a centralized control law, and the leader is a single point of failure for the formation. Hence, various decentralized control approaches (Carpenter 2002; Belanger et al. 2006; Lavaei et al. 2008; Chung et al. 2009; Ren and Beard 2004; Ren 2007; Gurfil and Mishne 2007; Smith and Hadaegh 2005; Min et al. 2011; Ramirez-Riberos et al. 2010; Wu et al. 2012) were proposed for formation keeping.

Carpenter (2002) proposed a decentralized linear-quadratic-Gaussian controller for autonomous establishment and maintenance of satellite formations. Two extended decentralized linear-quadratic-Gaussian controllers, each employing a unique information exchange scheme, were considered in Belanger et al. (2006). One of these algorithms employs a communication paradigm in which control and local state estimate information is exchanged between members of a formation after any one of the members executes a control. The other algorithm requires only that control information be exchanged between members. Lavaei et al. (2008) proposed a decentralized model predictive control for satellite formation. This decentralized controller consists of local estimators inherently, so that each local controller

estimates the state of the whole formation. Necessary and sufficient conditions for the stability of the formation under the proposed decentralized controller are obtained. A decentralized translational tracking control law based on oscillator phase synchronization is presented for satellite formation in Chung et al. (2009). The nonlinear relative motion dynamics is described in the Lagrangian form. The distributed stability and performance analysis tool, called contraction analysis, is used to yield exact nonlinear stability proofs. A decentralized implementation of a virtual structure coordination strategy is presented for formation control of deep-space spacecraft formation flying in Ren and Beard (2004). However, Ren and Beard (2004) require a bidirectional ring communication topology, which is rather restrictive in the sense that each satellite needs to explicitly identify its two adjacent neighbors in the group to form the ring. This approach is improved in Ren (2007), in which the constraint on the communication topology is relaxed to be having a directed spanning tree.

However, the above-cited decentralized control approaches have not considered the measurement hardware required for relative navigation in formation flying missions. It was assumed that the relative position and velocity of every satellite with respect to the virtual reference orbit is available in these studies. Some of these studies suggested the use of GPS for relative navigation of near-Earth formation flying missions (Busse and How 2002). However, GPS is not suitable for precision relative navigation in formation flying missions beyond low Earth orbits. Also, full collaboration and communication architecture among the formation satellites is not preferable for some formation flying missions due to high operational cost and technology immaturity. These two reasons motivate us to find an alternative (such as local relative measurements) to GPS-based relative navigation for formation flying missions.

Gurfil and Mishine (2007) developed a relative motion control law based on line-of-sight (LOS) measurements only (local relative measurements). Each satellite tracks its neighboring satellite to produce a LOS vector measurement, and the last satellite in the formation will track the first satellite to create a cyclic formation. However, bounded formation, which is obtained under the condition that the energies of satellites in the formation are matched, is not sufficient to satisfy all the requirements for satellite formation flying missions. In some missions, the relative positions and velocities between each of the satellites need to be controlled precisely.

Smith and Hadaegh (2005) presented a formation control design approach for deep-space spacecraft formation flying, based on local relative measurements. This approach exploits the redundancy inherent in a relative position specification to develop a family of equivalent control topologies for formation control. Switching between topologies provides redundancy and allows combinations of measured relative positions and communicated estimates to be used in the implementation of the formation control.

As mentioned earlier, fuel consumption is a major concern for satellite formation keeping mission. However, fuel consumption was not taken into account in the above two local relative measurements-based decentralized formation keeping methods.

5.2 Real-Time Optimal Formation Keeping in Leader–Follower Frame

Formation will drift apart slowly due to various environmental perturbations, such as gravitational perturbation and atmospheric drag, after the desired formation is established. Thus, formation keeping strategy is required. This section proposes a formation keeping strategy for satellite formation in eccentric orbits in a leader–follower frame (Wu et al. 2009, 2016). In the leader–follower frame, the leader is assumed to be well controlled, and the follower needs to track the desired relative motion with respect to the leader. This section proposes a real-time fuel-optimal control approach for the follower to track the leader using continuous low-thrust propulsion system. The control acceleration constraints are included in the optimization problem to avoid control saturation for low-thrust propulsion system. The fuel-optimal control problem is then converted into a quadratic programming problem by application of Legendre pseudospectral method. To solve the smooth optimal control problem, Legendre pseudospectral method has two advantages over other numerical methods: improved accuracy and faster convergence rate which makes real-time optimal control feasible. The proposed control law is simulated using MATLAB. The reported simulation results confirm the effectiveness of the proposed method.

5.2.1 Real-Time Optimal Control Law Design

In this subsection, a real-time optimal control is designed for satellite formation keeping. This proposed optimal control has potential real-time implementation. The cost function of the real-time fuel-optimal control for satellite formation keeping is chosen as follows.

$$J = \frac{1}{2} \int_{t_0}^{t_f} \mathbf{u}^T(t) \mathbf{u}(t) dt \quad (5.1)$$

where $\mathbf{u}(t) \in \mathbb{R}^3$ is the control acceleration, t_0 and t_f are the initial, final time of the optimization, respectively.

Remark 5.1 The chosen form of performance index is appropriate for power-limited, low-thrust propulsion. For variable specific impulse (I_{sp}) power-limited propulsion systems, the fuel consumption is directly proportional to the integral sum of squares of the control accelerations given above.

The fuel-optimal control problem for satellite formation keeping should be subject to the following constraints.

A. Relative dynamics constraints

In order to make the real-time fuel-optimal control approach practical, a linear optimal control problem should be formulated. Thus, a linearized relative motion model (2.64) considering J_2 perturbation and eccentricity is used in this section. The reference relative motion dynamics can be written as the following matrix form.

$$\dot{\mathbf{x}}_d(t) = A(t)\mathbf{x}_d(t) \quad (5.2)$$

where $\mathbf{x}_d \in \mathbb{R}^6$ denotes the reference relative state, $A(t)$ is the state matrix.

Remark 5.2 The desired formation geometry can be designed by choosing the initial reference relative state $\mathbf{x}_d(t_0)$. For instance, a circular formation can be designed as follows (Sabot et al. 2001):

$$\begin{cases} x_d(t_0) = (R/2) \cos \vartheta, & \dot{x}_d(t_0) = (-Rn/2) \sin \vartheta \\ y_d(t_0) = 2\dot{x}_0/n, & z_d(t_0) = \pm\sqrt{3}x_d(t_0), \quad \dot{z}_d(t_0) = \pm\sqrt{3}\dot{x}_d(t_0) \end{cases} \quad (5.3)$$

and

$$\frac{\dot{y}_d(t_0)}{x_d(t_0)} = -\frac{n(2+e)}{(1+e)^{1/2}(1-e)^{3/2}} \quad (5.4)$$

where R is the radius of the formation, ϑ is initial phasing angle of the formation, (5.4) is the periodicity or no-drift condition (Inalhan et al. 2002).

Considering (5.2) and (2.64) yields the following relative state error dynamics

$$\delta\dot{\mathbf{x}}(t) = A(t)\delta\mathbf{x}(t) + B(t)\mathbf{u}(t) \quad (5.5)$$

where $\delta\mathbf{x} = \mathbf{x} - \mathbf{x}_d$ denotes the relative state error. $\mathbf{x} \in \mathbb{R}^6$, $\mathbf{x}_d \in \mathbb{R}^6$ denote, respectively, the measured, reference relative state of the follower with respect to the leader. $\mathbf{u}(t)$ is the control acceleration vector.

B. Control acceleration constraints

The control acceleration constraints should be enforced, because continuous low-thrust engine for satellite formation keeping can only generate limited output force (Pencil et al. 2004).

$$\begin{aligned} -u_x^{\max} \leq u_x(t) \leq u_x^{\max}, \quad -u_y^{\max} \leq u_y(t) \leq u_y^{\max}, \quad -u_z^{\max} \leq u_z(t) \leq u_z^{\max}, \\ \forall t \in [t_0, t_f] \end{aligned} \quad (5.6)$$

where $u_x^{\max}, u_y^{\max}, u_z^{\max}$ denotes the limits of control accelerations $\mathbf{u}$ in three directions in LVLH frame. For simplicity, (5.6) is written as the following vector form

$$-\mathbf{u}_{\max} \leq \mathbf{u}(t) \leq \mathbf{u}_{\max} \quad (5.7)$$

C. Initial condition constraints

$$\delta \mathbf{x}(t_0) = \mathbf{x}_d(t_0) - \mathbf{x}(t_0); \quad (5.8)$$

D. Final condition constraints

$$\delta \mathbf{x}(t_f) = 0 \quad (5.9)$$

The above optimal control problem can be summarized. The problem is to determine the control acceleration $\mathbf{u}(t)$ and the corresponding state trajectory $\delta \mathbf{x}(t)$ to minimize the following cost function

$$J = \frac{1}{2} \int_{t_0}^{t_f} \mathbf{u}^T(t) \mathbf{u}(t) dt \quad (5.10)$$

subject to

$$\begin{cases} \delta \dot{\mathbf{x}}(t) = A(t) \delta \mathbf{x}(t) + B(t) \mathbf{u}(t) \\ -\mathbf{u}_{\max} \leq \mathbf{u}(t) \leq \mathbf{u}_{\max}, \quad \forall t \in [t_0, t_f] \\ \delta \mathbf{x}(t_0) = \mathbf{x}_d(t_0) - \mathbf{x}(t_0) \\ \delta \mathbf{x}(t_f) = 0 \end{cases} \quad (5.11)$$

The above optimal control can now be discretized into quadratic programming using Legendre pseudospectral method.

The implementation procedure for real-time optimal control is as follows:

- A. Obtain the measured relative state for the time t_0 , and calculate the relative state error between desired relative state and measured relative state.
- B. Solve the above linear optimization problem (5.10) and (5.11).
- C. Apply the control input $\mathbf{u}^*(t) \ t \in [t_0, t_0 + \Delta t]$ to system
- D. Set $t_0 = t_0 + \Delta t$, $t_f = t_f + \Delta t$, and go to the first step, repeat steps A–D.

Note that the stability of the above real-time optimal control approach can be ensured by the terminal constraint (5.9) according to Mayne et al. (2000).

5.2.2 Discretization Using Legendre Pseudospectral Method

The above optimal control can now be discretized into quadratic programming using Legendre pseudospectral method introduced in Sect. 4.4.

Because the above optimal control problem is formulated over the time interval $[t_0, t_f]$, and the LGL points lie in the interval $[-1, 1]$, the following transformation is used to express the problem for $\tau \in [\tau_0, \tau_N] = [-1, 1]$:

$$t = \frac{(t_f - t_0)\tau + (t_f + t_0)}{2} \quad (5.12)$$

It follows that (5.10) and (5.11) can be replaced by

$$\begin{cases} J = \frac{t_f - t_0}{4} \int_{-1}^1 (\mathbf{u}(\tau)^T \mathbf{u}(\tau)) d\tau \\ \delta \dot{\mathbf{x}}(\tau) = \frac{t_f - t_0}{4} [A(\tau) \delta \mathbf{x}(\tau) + B(\tau) \mathbf{u}(\tau)] \\ \delta \mathbf{x}(-1) = \mathbf{x}(t_0) - \mathbf{x}_d(t_0) \\ \delta \mathbf{x}(1) = 0, \quad -\mathbf{u}_{\max} \leq \mathbf{u}(\tau) \leq \mathbf{u}_{\max} \end{cases} \quad (5.13)$$

Note that the above functions are continuously differentiable with respect to their arguments, which are required in the Legendre pseudospectral method.

In summary, the fuel-optimal control problem for satellite formation keeping can be converted into the following mathematical programming problem using Legendre pseudospectral method:

Find the $9(N + 1)$ vector $\mathbf{X}_{\text{opt}}$

$$\mathbf{X}_{\text{opt}} = [\delta \mathbf{x}_0, \delta \mathbf{x}_1, \dots, \delta \mathbf{x}_N; \mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_N] \quad (5.14)$$

to minimize

$$J(\mathbf{X}_{\text{opt}}) = \frac{t_f - t_0}{4} \sum_{k=0}^N (\mathbf{u}(\tau_k)^T \mathbf{u}(\tau_k)) \omega_k \quad (5.15)$$

subject to

$$\begin{cases} \sum_{l=0}^N D_{kl} \delta \mathbf{x}_l - \frac{t_f - t_0}{2} [A(\tau_k) \delta \mathbf{x}(\tau_k) + B(\tau_k) \mathbf{u}(\tau_k)] = 0 \\ \mathbf{u}_{\min} \leq \mathbf{u}(\tau_k) \leq \mathbf{u}_{\max} \\ \delta \mathbf{x}(-1) = \mathbf{x}(t_0) - \mathbf{x}_d(t_0) \\ \mathbf{x}(1) = 0 \end{cases} \quad (5.16)$$

where ω_k are the weights given by

$$\omega_k := \{2/[N(N + 1)]\} \left\{ 1/[L_N(\tau_k)]^2 \right\}, \quad k = 0, 1, \dots, N \quad (5.17)$$

For the details of Legendre pseudospectral method, please refer to Smith and Hadaegh (2005).

5.2.3 Computational Considerations of Quadratic Programming Problem

This section discusses issues related to the implementation and solution of the optimization problem presented in the preceding sections. The methods presented here do not change the basic constraints already described, but assist greatly in obtaining solutions within a practical and reasonable computational time interval.

As mentioned in previous chapter, scaling a mathematical programming problem is very important in order to obtain robust and rapid convergence to the final solution. One way to construct a well-scaled problem is to normalize the independent variables to have the same range. For example, $0 \leq \tilde{x}_k \leq 1$, where $\tilde{x}_k$ denotes the scaled variables. So the following normalizations are used in this work.

$$\delta\tilde{x} = \delta x / \bar{d}_{\max}, \quad \delta\tilde{y} = \delta y / \bar{d}_{\max}, \quad \delta\tilde{z} = \delta z / \bar{d}_{\max} \quad (5.18)$$

$$\delta\tilde{\dot{x}} = \delta\dot{x} / v_{\max}, \quad \delta\tilde{\dot{y}} = \delta\dot{y} / v_{\max}, \quad \delta\tilde{\dot{z}} = \delta\dot{z} / v_{\max} \quad (5.19)$$

$$\tilde{u}_x = 10^5 u_x, \quad \tilde{u}_y = 10^5 u_y, \quad \tilde{u}_{j,z} = 10^5 u_z \quad (5.20)$$

where $\bar{d}_{\max}$ and $v_{\max}$ denote the possible maximal relative position and velocity, respectively.

The Legendre pseudospectral method has a larger radius of convergence than other numerical methods, and it may not require a good initial guess for convergence. However, an educated initial guess does improve convergence rate and robustness. In this section, an initial guess is generated by integrating the relative dynamics numerically using zero control acceleration.

The quadratic programming problem is solved by the command “quadprog” in MATLAB.

5.2.4 Numerical Simulations

In this section, simulations are performed using MATLAB to validate the proposed formation keeping approaches. The chief satellite selected for the numerical simulations has the following orbital elements

$$a = 7100 \text{ km}, \quad e = 0.01, \quad i = 45^\circ, \quad \varpi = 20^\circ, \quad \Omega = 0^\circ, \quad \theta = 340^\circ \quad (5.21)$$

A full nonlinear propagator with the Earth zonal harmonic perturbations up to J_{21} , and atmospheric drag in a software “Orbital Mechanics with MATLAB” (<http://www.cdeagle.com/html/ommatlab.html>, June, 2010) is used for simulation. The area-to-mass ratio of two satellites is assumed to be identical. The number of LGL points is chosen as 20. More accurate results can be obtained in this optimization problem with more LGL points at the expense of computational cost. The clock frequency of the CPU used is 1.86 GHz. The optimization can be completed within 0.03 s, which promises great potential for real-time computations.

The first example in this section is to correct initial position errors by using real-time fuel-optimal control. The prediction horizon is chosen as $t_f - t_0 = 3000$ s. The assumed maximum control accelerations in three directions in LVLH frame are identical and chosen to be $u_{\max} = 3 \times 10^{-4} \text{ m/s}^2$. The initial relative velocity errors are assumed to be zeros. The initial relative position errors are assumed to be

$$\delta x_0 = 100 \text{ m}, \quad \delta y_0 = -200 \text{ m}, \quad \delta z_0 = 100 \text{ m} \quad (5.22)$$

The time histories of the relative position error, velocity error, and control accelerations are shown in Figs. 5.1, 5.2, and 5.3, respectively. As seen from Figs. 5.1, 5.2, and 5.3, the position error and velocity error are corrected in about one orbital period using fuel-optimal control, and the control accelerations are kept within their limits.

In the second example, real-time optimal control is used to maintain formation against various perturbations over five orbital periods. The initial relative state error is assumed to be zero. The prediction horizon is chosen as $t_f - t_0 = 3000$ s. The time histories of the relative position error, velocity error, and control accelerations are shown in Figs. 5.4, 5.5, and 5.6, respectively.

Fig. 5.1 Relative position error under real-time optimal control

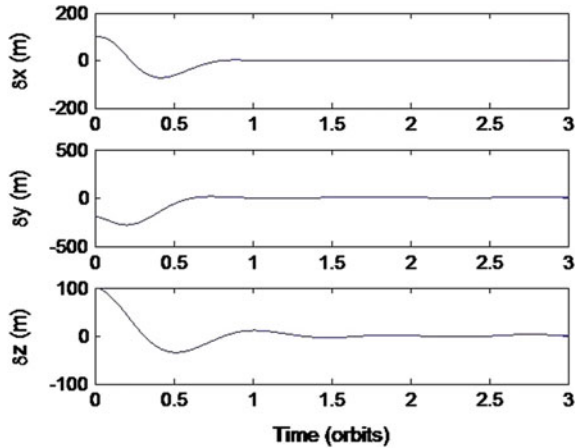


Fig. 5.2 Relative velocity error under real-time optimal control

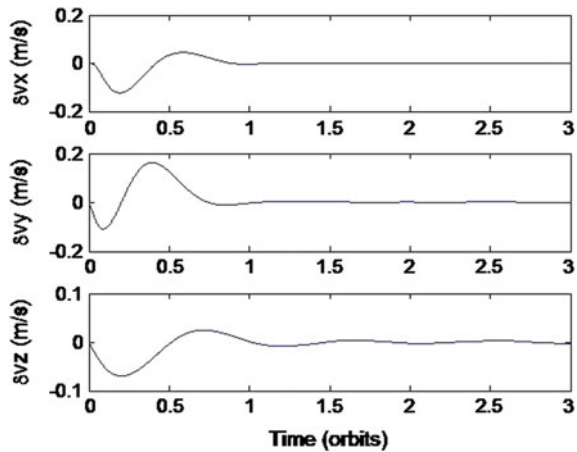
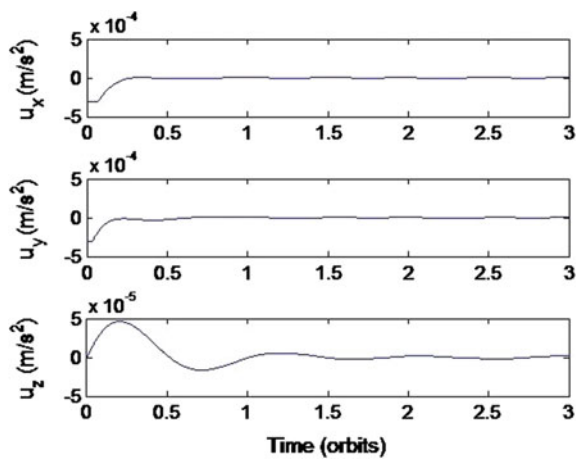


Fig. 5.3 Control acceleration under real-time optimal control



The choice of the prediction horizon $t_f - t_0$ is a trade-off between fuel consumption and control accuracy. Suppose the control accuracy needs to be increased, the prediction horizon can be decreased. Thus, in the third example, the prediction horizon is chosen as $t_f - t_0 = 600$ s. The time histories of the relative position error, velocity error, and control accelerations are shown in Figs. 5.7, 5.8 and 5.9, respectively. As observed from simulation results, the relative state error in the third example is much less than that of the previous example.

Fig. 5.4 Relative position error under real-time optimal control with $t_f - t_0 = 3000$ s

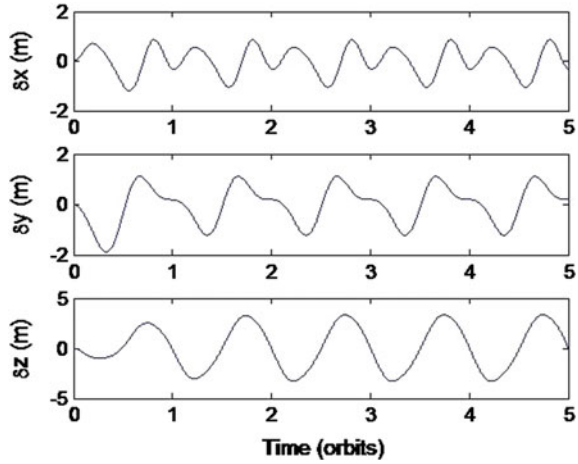
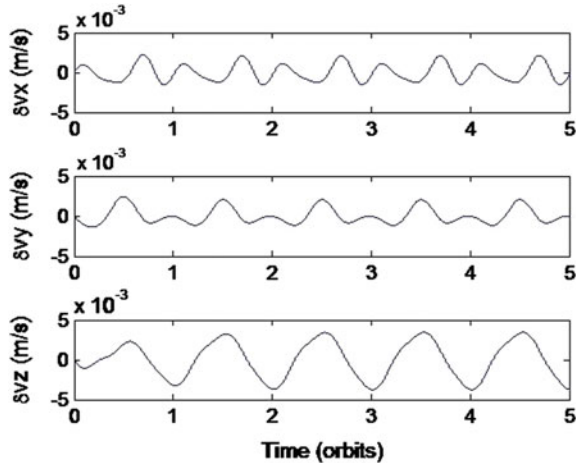


Fig. 5.5 Relative velocity error under real-time optimal control with $t_f - t_0 = 3000$ s



5.3 Decentralized Formation Control Using Local Relative Measurements

This section introduces a decentralized control design technique for satellite formation in elliptic orbits using algebraic graph theories (Wu et al. 2010). Each satellite controls itself by using only the relative states describing its position and velocity relative to neighboring satellites it can sense. These relative states can be obtained by local relative measurements, such as LOS measurements. The formation control problem with n satellites is thus reduced into n stabilization problems of single satellite with the identical relative dynamics differentiated by a distinct scalar parameter. This scalar parameter takes the values of nonzero eigenvalues of a Laplacian matrix which represents the topology of relative measurements among all

Fig. 5.6 Control acceleration under real-time optimal control with $t_f - t_0 = 3000$ s

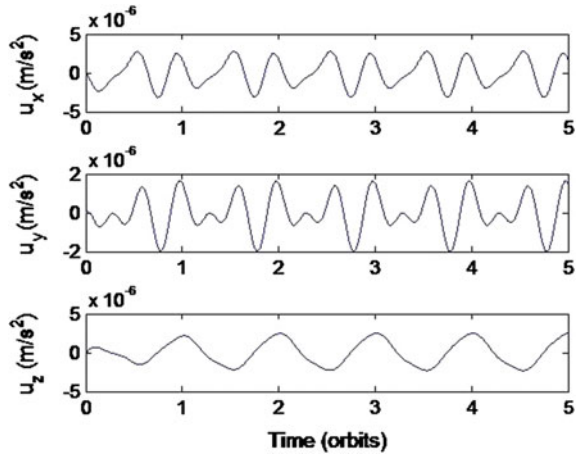
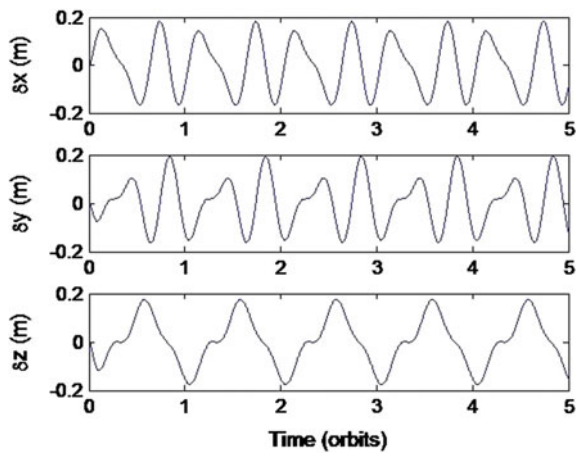


Fig. 5.7 Relative position error under real-time optimal control with $t_f - t_0 = 600$ s



the satellites. The resulting relative dynamics can be treated as a LTV system with uncertain parameters and an optimal guaranteed cost control law can be applied. Simulation results are presented to show the efficiency of the proposed decentralized formation control strategy for satellite formation flying.

5.3.1 Problem Formulation of Decentralized Formation Control

Consider a formation of n satellites with identical dynamics. The relative motion dynamics of each satellite with respect to an elliptic reference orbit is modeled in (2.83). Now it is rewritten with index as follows:

Fig. 5.8 Relative velocity error under real-time optimal control with $t_f - t_0 = 600$ s

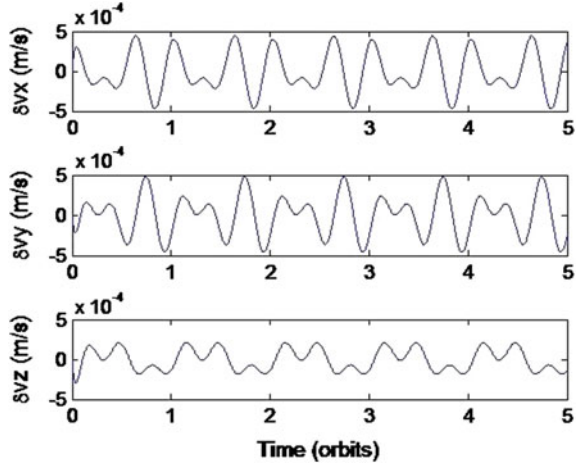
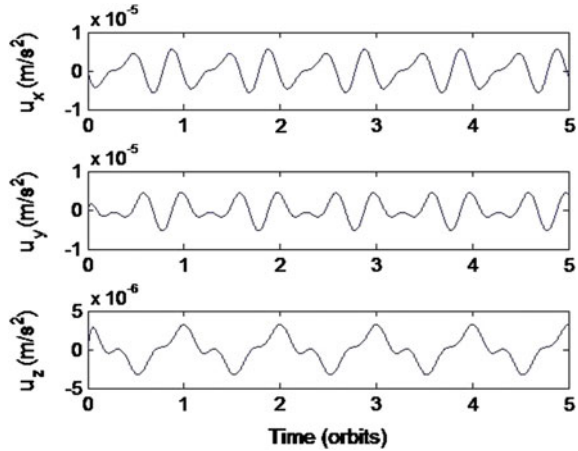


Fig. 5.9 Control input by using real-time optimal control with $t_f - t_0 = 600$ s



$$\mathbf{x}'_i(f) = A(f)\mathbf{x}_i(f) + B(f)\mathbf{u}_i(f), \quad \forall i = 1, 2, \dots, n \quad (5.23)$$

where the subscript i denotes the i th satellite in a formation.

Motivated by the study of cooperative control for multi-agent systems using local interaction (Fax and Murray 2004; Ren and Atkins 2007), this section proposes a decentralized formation control approach in which each satellite controls itself by using only relative states describing its position and velocity relative to neighboring satellites it can sense. These relative states can be obtained from local relative measurements, such as local LOS measurements.

The proposed decentralized formation control is to design the following decentralized state-feedback control for each satellite

$$\mathbf{u}_i(f) = \sigma_i K \sum_{j \in N_i} \left(\Delta \mathbf{x}_{ij}(f) - \Delta \mathbf{x}_{ij}^d(f) \right), \quad \forall i, j = 1, 2, \dots, n \quad (5.24)$$

where K is a constant state-feedback gain matrix. The set N_i represents the set of satellites which the i th satellite can sense. $\Delta \mathbf{x}_{ij} = \mathbf{x}_j - \mathbf{x}_i$ and $\Delta \mathbf{x}_{ij}^d = \mathbf{x}_j^d - \mathbf{x}_i^d$ denote, respectively, the actual and desired relative states of the j th with respect to the i th satellite. $\mathbf{x}_i$ and $\mathbf{x}_i^d$ denote, respectively, the actual and the desired relative states of the i th satellite with respect to the reference orbit.

In some formation flying missions, if one of the satellites runs out of fuel faster, the entire distributed system can be non-operational. Generally, this fuel imbalance may shorten total mission duration. Thus, besides fuel minimization, fuel balance should be considered to guarantee maximum mission lifetime. The parameters σ_i denote positive weight scalars for fuel balance among satellites. It can be chosen as given below

$$\sigma_i = \frac{\text{Fuel}_i}{\frac{1}{n} \sum_{j=1}^n \text{Fuel}_j}, \quad \forall i = 1, 2, \dots, n \quad (5.25)$$

where Fuel_i denotes the fuel left on the i th satellite.

The goal of formation control is

$$\lim_{f \rightarrow \infty} \left(\Delta \mathbf{x}_{ij}(f) - \Delta \mathbf{x}_{ij}^d(f) \right) = 0, \quad \forall i, j = 1, 2, \dots, n \quad (5.26)$$

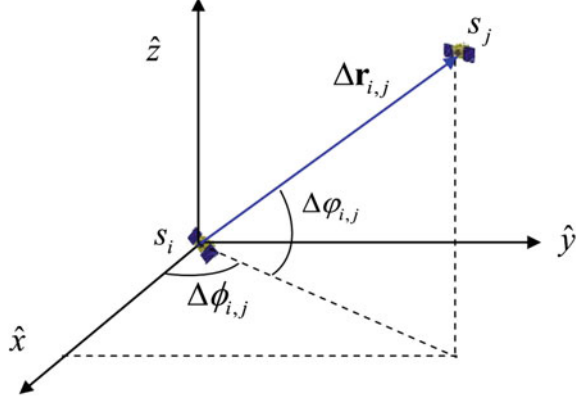
As observed from the above equation, the desired relative positions and velocities between each pair of the satellites are obtained. Note that the relative states $\Delta \mathbf{x}_{ij}$ between satellites in formation need to be controlled, but not necessarily the relative states $\mathbf{x}_i$ of satellites with respect to the reference orbit.

Assumption 5.1 The desired relative states between satellites $\Delta \mathbf{x}_{ij}^d$ are assumed to be also governed by the relative motion dynamics in (2.83) without control inputs as follows.

$$\Delta \dot{\mathbf{x}}_{ij}^d(f) = A(f) \Delta \mathbf{x}_{ij}^d(f) \quad \forall i, j = 1, 2, \dots, n \quad (5.27)$$

Remark 5.3 The relative state between satellites $\Delta \mathbf{x}_{ij} \forall i, j = 1, 2, \dots, n$ can be obtained by local relative measurements, such as local LOS measurement. LOS measurements, as illustrated in Fig. 5.10, include relative distance measurement and relative orientation measurements in (5.28). LOS measurements can be acquired by several readily available methods, such as the use of vision-based sensors that combine star trackers and optical/RF ranging (Alonso et al. 2000) or a combination of onboard optical tracking with rate gyros for the determination of LOS. In any case, only the relative LOS vector is required for the control and no absolute position of each satellite is required.

Fig. 5.10 Illustration of LOS measurements among satellites



$$\begin{cases} \Delta r_{i,j} = \sqrt{\Delta x_{i,j}^2 + \Delta y_{i,j}^2 + \Delta z_{i,j}^2} \\ \Delta \varphi_{i,j} = \arcsin(\Delta z_{i,j} / \Delta r_{i,j}) \\ \Delta \phi_{i,j} = \arctan(\Delta y_{i,j} / \Delta x_{i,j}) \quad \forall i, j = 1, 2, \dots, n. \end{cases} \quad (5.28)$$

5.3.2 Decentralized Formation Control Design

This section derives the condition under which the goal of formation control (5.26) is achieved. Meanwhile, the relative state of each satellite at steady state is given. The following lemma will be used to derive the proposed control law.

Lemma 5.1 (Horn and Johnson 1991) Suppose that $M \in \mathbb{R}^{m \times m}$, $N \in \mathbb{R}^{n \times n}$, $X \in \mathbb{R}^{m \times m}$ and $Y \in \mathbb{R}^{n \times n}$. The following results hold:

- (i) $(M \otimes N)(X \otimes Y) = MX \otimes NY$, where $\otimes$ denotes the Kronecker product;
- (ii) Suppose that M and N are invertible. Then $(M \otimes N)^{-1} = M^{-1} \otimes N^{-1}$; and
- (iii) Let $\lambda_1, \dots, \lambda_m$ be the eigenvalues of M and $\mu_1, \dots, \mu_n$ be those of N . Then the eigenvalues of $M \otimes N$ are $\lambda_i \mu_j$, $i = 1, \dots, m$, $j = 1, \dots, n$.

The decentralized formation control law given in (5.24) can be rewritten as

$$\mathbf{u}_i(f) = \sigma_i K \sum_{j \in N_i} \left[(\mathbf{x}_j(f) - \mathbf{x}_i(f)) - (\mathbf{x}_j^d(f) - \mathbf{x}_i^d(f)) \right], \quad \forall i, j \in [1, 2, \dots, n] \quad (5.29)$$

where $\mathbf{x}_i$ and $\mathbf{x}_i^d$ denote, respectively, the actual and the desired relative states of the i th satellite with respect to reference orbit.

Collecting the equations for all satellites into a single system using Kronecker product, the system in (5.23) and (5.24) is represented as follows:

$$\mathbf{x}'(f) = (I_n \otimes A(f))\mathbf{x}(f) + (I_n \otimes B(f))\mathbf{u}(f) \quad (5.30)$$

$$\mathbf{u}(f) = (I_n \otimes K)(L \otimes I_6)(\mathbf{x}^d(f) - \mathbf{x}(f)) \quad (5.31)$$

where

$$\mathbf{x} = [\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n]^T, \quad \mathbf{x}^d = [\mathbf{x}_1^d, \mathbf{x}_2^d, \dots, \mathbf{x}_n^d]^T, \quad \mathbf{u} = [\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n]^T$$

and L is a Laplacian matrix presenting the topology of relative measurements among satellites in a formation and weight on each satellite for fuel balance, as defined in (A.3).

Theorem 5.1 Consider a satellite formation described by (5.23) with the control law in (5.24). Suppose Assumption 5.1 is valid and if the control gain K in (5.24) can be chosen such that the following system is stable

$$\chi'(f) = (A(f) - \lambda_i B(f)K)\chi(f) \quad \forall i = 2, 3, \dots, n \quad (5.32)$$

where χ are the state variable of appropriate dimensions. λ_i are nonzero eigenvalues of L , the goal of formation control defined in (5.26) is achieved if and only if the graph G_n representing the topology of the relative measurement among satellite in the formation has a directed spanning tree.

Furthermore, the relative state of each satellite at steady state is given by, with $\mathbf{x}_i^e = \mathbf{x}_i^d - \mathbf{x}_i$

$$\mathbf{x}_{i,ss}(f) = \mathbf{x}_i^d(f) - \left(\sum_{i=1}^n v_i \Phi_A(f) \right) \mathbf{x}_i^e(0), \quad \forall i = 1, 2, \dots, n \quad (5.33)$$

where $[v_1, v_2, \dots, v_n]$ is the associated left eigenvector of L corresponding to zero eigenvalue, and $\sum_{i=1}^n v_i = 1$, $v_i \geq 0$. $\Phi_A(f)$ is the state transition matrix of the system in (5.23) without control input.

Proof Note that the systems described by (5.23) with the control law in (5.24) can be represented in a single system in (5.30) with control law in (5.31) using the Kronecker product. Substituting (5.31) into (5.30) yields

$$\mathbf{x}'(f) = (I_n \otimes A(f))\mathbf{x}(f) + (I_n \otimes B(f)K)(L \otimes I_6)(\mathbf{x}^d(f) - \mathbf{x}(f)) \quad (5.34)$$

Assumption 5.1 implies

$$\mathbf{x}^{td}(f) = (I_n \otimes A(f))\mathbf{x}^d(f) \quad (5.35)$$

Define

$$\mathbf{x}^e(f) = \mathbf{x}^d(f) - \mathbf{x}(f) \quad (5.36)$$

Subtracting (5.35) from (5.34) and using the definition in the above equation lead to

$$\mathbf{x}'^e(f) = (I_n \otimes A(f))\mathbf{x}^e(f) - (I_n \otimes B(f)K)(L \otimes I_6)\mathbf{x}^e(f) \quad (5.37)$$

Let $T \in C_{n \times n}$ be Schur transformation of L , meaning the unitary matrix, such that $U = T^{-1}LT$ is upper triangular with the eigenvalues of L along the diagonal. Obviously, $T \otimes I_6$ can transform $L \otimes I_6$ into $U \otimes I_6$ using Kronecker product's property $(A \otimes B)(C \otimes D) = AC \otimes BD$.

$$(T \otimes I_6)^{-1}(L \otimes I_6)(T \otimes I_6) = U \otimes I_6 \quad (5.38)$$

Let $\tilde{\mathbf{x}} = (T \otimes I_6)\mathbf{x}$, (5.37) is transformed into

$$\tilde{\mathbf{x}}'^e(f) = (I_n \otimes A(f))\tilde{\mathbf{x}}^e(f) - [U \otimes (B(f)K)]\tilde{\mathbf{x}}^e(f) \quad (5.39)$$

Note that $I_n \otimes A(f)$ is block diagonal matrix and $U \otimes B(f)K$ is block upper triangular matrix or block diagonal matrix with the diagonal block $\lambda_i B(f)K$, so the stability of the above complex systems is equivalent to the stability of the n sub-systems in the diagonal blocks. Its n diagonal subsystems are of the form:

$$\tilde{\mathbf{x}}_i'^e(f) = A(\theta)\tilde{\mathbf{x}}_i^e(f) - \lambda_i B(f)K\tilde{\mathbf{x}}_i^e(f), \quad i = 1, 2, \dots, n \quad (5.40)$$

According to Proposition A.4, if and only if the topology graph representing the relative measurements between satellites has a spanning tree, then zero is an eigenvalue of algebraic multiplicity one for the Laplacian L . So the Laplacian L can be transformed into the following form:

$$T^{-1}LT = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & \vdots & \vdots & \vdots \\ \vdots & \vdots & \hat{U}_{(n-1) \times (n-1)} & \vdots \\ 0 & \vdots & \vdots & \vdots \end{bmatrix} \quad (5.41)$$

where $\hat{U}_{(n-1) \times (n-1)}$ is upper matrix with the diagonal entries of λ_i , $i = 2, 3, \dots, n$.

By using (5.41), the transformed system in (5.39) can be rewritten as

$$\begin{aligned}\tilde{\mathbf{x}}^{fe}(f) &= \left(\begin{bmatrix} A(f) & 0_{6 \times 6} & \cdots & 0_{6 \times 6} \\ 0_{6 \times 6} & A(f) & & \\ \vdots & & \ddots & \\ 0_{6 \times 6} & & & A(f) \end{bmatrix} - \begin{bmatrix} 0_{6 \times 6} & 0_{6 \times 6} & \cdots & 0_{6 \times 6} \\ 0_{6 \times 6} & & & \\ \vdots & & \hat{U} \otimes (B(f)K) & \\ 0_{6 \times 6} & & & \end{bmatrix} \right) \tilde{\mathbf{x}}^e(f) \\ &= \begin{bmatrix} A(f) & 0_{6 \times 6} & \cdots & 0_{6 \times 6} \\ 0_{6 \times 6} & & & \\ \vdots & & \mathbf{U} & \\ 0_{6 \times 6} & & & \end{bmatrix} \tilde{\mathbf{x}}^e(f)\end{aligned}\quad (5.42)$$

where $\mathbf{U} = I_{n-1} \otimes A - \hat{U} \otimes (BK)$.

The state transition matrix of transformed system in (5.42) can be written as

$$\Phi_{\tilde{\mathbf{x}}^e}(f) = \text{diag}(\Phi_A(f), \Phi_u(f)) \quad (5.43)$$

If the control gain K in (5.24) can be chosen such that the closed-loop system in (5.32) is stable, then

$$\lim_{f \rightarrow \infty} \Phi_u(f) = 0 \quad (5.44)$$

According to Proposition A.4, it is known that the first column of the unitary matrix T is $[1, 1, \dots, 1]_n^T$ and the first row of the matrix T^{-1} is $[v_1, v_2, \dots, v_n]$. Transforming the state variable $\tilde{\mathbf{x}}^e$ back into the original system yields

$$\begin{aligned}\mathbf{x}_{ss}^e(f) &= (T \otimes I_6) \lim_{t \rightarrow \infty} \Phi_{\tilde{\mathbf{x}}^e}(f) (T \otimes I_6)^{-1} \mathbf{x}^e(0) \\ &= (T \otimes I_6) (\text{diag}(\Phi_A(f), 0, \dots, 0)) (T \otimes I_6)^{-1} \mathbf{x}^e(0) \\ &= ([1, 1, \dots, 1]_n^T \otimes I_6) \sum_{i=1}^n (v_i \Phi_A(f) \mathbf{x}_i^e(0))\end{aligned}\quad (5.45)$$

where the subscript ss denotes steady-state variable.

That is

$$\mathbf{x}_{i,ss}^e(f) = \sum_{i=1}^n (v_i \Phi_A(f) \mathbf{x}_i^e(0)), \quad i = 1, 2, \dots, n \quad (5.46)$$

where $\sum_{i=1}^n v_i = 1$, and $v_i \geq 0$, $i = 1, 2, \dots, n$.

So the relative state of each satellite at steady states is

$$\mathbf{x}_{i,ss}(f) = \mathbf{x}_i^d(f) - \mathbf{x}_{i,ss}^e(f) = \mathbf{x}_i^d(f) - \sum_{i=1}^n (v_i \Phi_A(f) \mathbf{x}_i^e(0)), \quad i = 1, 2, \dots, n \quad (5.47)$$

Using the above equation leads to

$$\begin{aligned} \lim_{f \rightarrow \infty} \Delta \mathbf{x}_{i,j}(f) &= \mathbf{x}_{j,ss}(f) - \mathbf{x}_{i,ss}(f) \\ &= \left(\mathbf{x}_j^d(f) - \mathbf{x}_{j,ss}^e(f) \right) - \left(\mathbf{x}_i^d(f) - \mathbf{x}_{i,ss}^e(f) \right) \\ &= \mathbf{x}_j^d(f) - \mathbf{x}_i^d(f) \\ &= \Delta \mathbf{x}_{i,j}^d(f), \quad \forall i, j = 1, 2, \dots, n. \end{aligned} \quad (5.48)$$

where $\Delta \mathbf{x}_{j,i}^d(f)$ is the desired relative state of the i th with respect to the j th satellite. As concluded from (5.48), the goal of the formation defined in (5.26) is achieved. $\square$

Remark 5.4 Observe that if there is a satellite i in the formation which cannot measure other satellite, then there will be no control input acting on this satellite. So the relative state of the i th satellite with respect to the reference orbit is obtained as $\mathbf{x}_i(f) = \Phi_A(f) \mathbf{x}_i(0)$. Therefore, the other satellites will converge to $\mathbf{x}_i(f)$. That is $v_i = 1$ and $v_k = 0, \forall k \neq i$.

In general, the initial state of a satellite contributes to the steady-state value if and only if the satellite has directed path to all the other satellites in the formation. Thus, $v_i \neq 0$ for any satellite which has directed paths to all the other satellites, and $v_i = 0$ otherwise. As a special case, the initial state of each satellite in a formation contributes to steady-state value if and only if the graph is strongly connected. The above argument can be explained as follows. If there is no path from the j th to the m th satellite in the formation, it is impossible for $\mathbf{x}_m(f)$ to be influenced by $\mathbf{x}_j(f)$. On the other hand, if there is a path from the j th satellite to every other satellite, then $\mathbf{x}_i(f) \forall i \neq j$, will be influenced by $\mathbf{x}_j(f)$.

Next, the problem is to design state-feedback gain K such that the system in (5.32) is stable. The closed-loop system in (5.32) can be viewed as the following linear parameter uncertain system with state-feedback control.

$$\begin{cases} \boldsymbol{\chi}'(f) = A(f) \boldsymbol{\chi}(f) - \lambda_i B(f) \mathbf{v}(f); \\ \mathbf{v}(f) = K \boldsymbol{\chi}(f), \quad \forall i = 2, 3, \dots, n \end{cases} \quad (5.49)$$

The optimal guaranteed cost control law to be introduced in the next section is chosen to design the above state-feedback controller, because it can stabilize the linear system (5.32) while minimizing a quadratic cost function. The choice of the weight matrices in the cost function is a trade-off between the fuel consumption and the convergence rate.

5.3.3 Optimal Guaranteed Cost Control Design

In this subsection, a state-feedback controller is designed using the optimal guaranteed cost control for the system in (5.32). The optimal guaranteed cost control law can minimize the quadratic cost function while stabilizing the linear norm-bounded uncertain system. The weight matrices in quadratic cost function are used to balance fuel consumption against convergence time. The optimal guaranteed cost control is reviewed in Appendix B.

Now the problem is making use of the optimal guaranteed cost control to design a state-feedback controller that can asymptotically stabilize the system in (5.32) for each nonzero eigenvalue λ_i of L . So the LTV system with an uncertain parameter as in (5.49) needs to be written into the linear norm-bounded uncertain system in (A.4). For mildly elliptic reference orbit ($e \leq 0.1$), it is valid for (5.49) to be written as linear norm-bounded uncertain system. For the case of $e > 0.1$, it will be conservative to express the time-varying part of the dynamics (5.49) as uncertainty. As seen from (2.83), the in-plane (x and y) and out-of-plane (z) motions are decoupled, so the controller for in plane and out-of-plane can be designed separately. The controller for in plane is designed firstly.

In order to rewrite time-varying system (5.49) into the linear norm-bounded uncertain system as in (A.4), the following variables are defined

$$\gamma = \frac{2e \sin f}{1 + e \cos f}, \quad \rho = \frac{e \cos f}{1 + e \cos f}, \quad \xi = \frac{1}{(1 + e \cos f)^4} \quad (5.50)$$

Then,

$$\gamma = \gamma_r \Delta_r, \quad \rho = \rho_c + \rho_r \Delta_\rho, \quad \xi = \xi_c + \xi_r \Delta_\xi, \quad \lambda = \lambda_c + \lambda_r \Delta_\lambda \quad (5.51)$$

where Δ_r , Δ_ρ , Δ_ξ , and Δ_λ are the uncertainties, γ_r , ρ_r , ξ_r , and λ_r are the known bounds of these uncertainties. ρ_c , ξ_c , and λ_c are the nominal values of ρ , ξ and λ , and

$$\rho_c = \frac{-e^2}{1 - e^2}, \quad \rho_r = \frac{e}{1 - e^2}, \quad |\Delta_r| \leq 1, \quad |\Delta_\rho| \leq 1, \quad |\Delta_\xi| \leq 1, \quad |\Delta_\lambda| \leq 1 \quad (5.52)$$

By using (5.50) and (5.51), the in-plane part of the system in (5.49) can be expressed as the following linear norm-bounded uncertain system:

$$\chi'_{\text{in}} f = (\bar{A}_{\text{in}} + \Delta \bar{A}_{\text{in}}) \chi_{\text{in}}(f) + (\bar{B}_{\text{in}} + \Delta \bar{B}_{\text{in}}) \mathbf{v}_{\text{in}}(f) \quad (5.53)$$

where $\bar{A}_{\text{in}}$ and $\Delta\bar{A}_{\text{in}}$ are nominal and uncertain parts of system matrix, $\bar{B}_{\text{in}}$ and $\Delta\bar{B}_{\text{in}}$ are nominal and uncertain parts of input matrix, and

$$\bar{A}_{\text{in}} = \begin{bmatrix} 0 & 3 - 2\rho_c & 2 & 0 \\ 1 & 0 & 0 & 0 \\ -2 & 0 & 0 & \rho_c \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \Delta\bar{A}_{\text{in}} = \begin{bmatrix} \gamma_r\Delta_r & -2\rho_r\Delta_\rho & 0 & -\gamma_r\Delta_r \\ 0 & 0 & 0 & 0 \\ 0 & \gamma_r\Delta_r & \gamma_r\Delta_r & \rho_r\Delta_\rho \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (5.54)$$

$$\bar{B}_{\text{in}} = \frac{-\lambda_c\check{\xi}_c(1-e^2)^3}{n^2} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad (5.55)$$

$$\Delta\bar{B}_{\text{in}} = \frac{(\lambda_r\check{\xi}_c\Delta_\lambda + \lambda_c\check{\xi}_r\Delta_\xi + \lambda_r\check{\xi}_r\Delta_\lambda\Delta_\xi)(1-e^2)^3}{n^2} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

and $\Delta\bar{A}_{\text{in}}$, $\Delta\bar{B}_{\text{in}}$ are written as the following form:

$$[\Delta\bar{A}_{\text{in}} \quad \Delta\bar{B}_{\text{in}}] = D_{\text{in}}F_{\text{in}}[E_{1,\text{in}} \quad E_{2,\text{in}}] \quad (5.56)$$

and

$$D_{\text{in}} = [D_{1,\text{in}} \quad D_{2,\text{in}}] \quad (5.57)$$

$$D_{1,\text{in}} = \begin{bmatrix} \gamma_r & -\gamma_r & 0 & 0 & -2\rho_r & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma_r & \gamma_r & 0 & \rho_r \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (5.58)$$

$$D_{2,\text{in}} = \frac{(1-e^2)^3}{n^2} \begin{bmatrix} \lambda_r\check{\xi}_c & 0 & \lambda_c\check{\xi}_r & 0 & \lambda_r\check{\xi}_r & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_r\check{\xi}_c & 0 & \lambda_c\check{\xi}_r & 0 & \lambda_r\check{\xi}_r \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (5.59)$$

$$E_{1,\text{in}} = \begin{bmatrix} \hat{E}_{1,\text{in}} \\ 0_{6 \times 4} \end{bmatrix}, \quad \hat{E}_{1,\text{in}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}^T \quad (5.60)$$

$$E_{2,\text{in}} = \begin{bmatrix} 0_{6 \times 2} \\ \hat{E}_{2,\text{in}} \end{bmatrix}, \quad \hat{E}_{2,\text{in}} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}^T \quad (5.61)$$

$$F_{\text{in}} = \text{diag}\{\Delta_\gamma, \Delta_\gamma, \Delta_\gamma, \Delta_\gamma, \Delta_\rho, \Delta_\rho, \Delta_\lambda, \Delta_\lambda, \Delta_\xi, \Delta_\xi, \Delta_\lambda \Delta_\xi, \Delta_\lambda \Delta_\xi\}, \quad F_{\text{in}}^T F_{\text{in}} \leq I, \quad (5.62)$$

with F_{in} denoting the uncertainty.

The out-of-plane part of the system in (5.49) can also be expressed as the following linear norm-bounded uncertain system.

$$\chi'_{\text{out}}(f) = (\bar{A}_{\text{out}} + \Delta \bar{A}_{\text{out}}) \chi_{\text{out}}(f) + (\bar{B}_{\text{out}} + \Delta \bar{B}_{\text{out}}) \mathbf{v}_{\text{out}}(f) \quad (5.63)$$

where

$$\bar{A}_{\text{out}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \Delta \bar{A}_{\text{out}} = \begin{bmatrix} \gamma_r \Delta_r & \rho_r \Delta_\rho \\ 0 & 0 \end{bmatrix} \quad (5.64)$$

$$\begin{aligned} \bar{B}_{\text{out}} &= \frac{-\lambda_c \xi_c (1 - e^2)^3}{n^2} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ \Delta \bar{B}_{\text{out}} &= \frac{(\lambda_r \xi_c \Delta_\lambda + \lambda_c \xi_r \Delta_\xi + \lambda_r \xi_r \Delta_\lambda \Delta_\xi)(1 - e^2)^3}{n^2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{aligned} \quad (5.65)$$

and $\Delta \bar{A}_{\text{out}}, \Delta \bar{B}_{\text{out}}$ are written as the following form:

$$\begin{bmatrix} \Delta \bar{A}_{\text{out}} & \Delta \bar{B}_{\text{out}} \end{bmatrix} = D_{\text{out}} F_{\text{out}} \begin{bmatrix} E_{1,\text{out}} & E_{2,\text{out}} \end{bmatrix} \quad (5.66)$$

and

$$\begin{aligned} D_{\text{out}} &= \begin{bmatrix} D_{1,\text{out}} & D_{2,\text{out}} \end{bmatrix}, \quad D_{1,\text{out}} = \begin{bmatrix} \gamma_r & \rho_r \\ 0 & 0 \end{bmatrix}, \\ D_{2,\text{out}} &= \frac{(1 - e^2)^3}{n^2} \begin{bmatrix} \lambda_r \xi_c & \lambda_c \xi_r & \lambda_r \xi_r \\ 0 & 0 & 0 \end{bmatrix} \end{aligned} \quad (5.67)$$

$$E_{1,\text{out}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}^T, \quad E_{2,\text{out}} = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \end{bmatrix}^T \quad (5.68)$$

$$F_{\text{out}} = \text{diag}\{\Delta_\gamma, \Delta_\rho, \Delta_\lambda, \Delta_\xi, \Delta_\lambda \Delta_\xi\}, \quad F_{\text{out}}^T F_{\text{out}} \leq I \quad (5.69)$$

Then, the state-feedback control gain for the norm-bounded uncertain systems in (5.53) and (5.63) can be designed using the optimal guaranteed cost control.

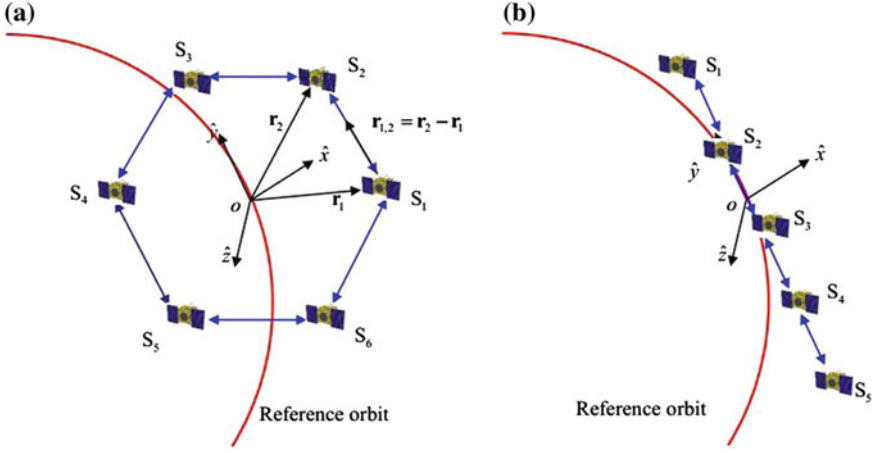


Fig. 5.11 Topologies of local relative measurements between formation satellites in LVLH frame. **a** Cyclic topology for circular formation. **b** Line topology for in-plane formation

5.3.4 Simulation Results

In this section, simulations using STK and MATLAB demonstrate the performance of the proposed decentralized formation control for satellite formation. For illustration, two scenarios of keeping a circular formation and an in-plane formation are considered. It is assumed that their corresponding relative measurement topologies are undirected cyclic and line topology, respectively, as shown in Fig. 5.11. In the process of simulation, the control accelerations are calculated in MATLAB using data transmitted from STK via STK connect. The motion of each satellite is propagated through STK real-time propagator with J_2 , J_3 , and J_4 perturbations.

5.3.4.1 Circular Formation

For the desired circular formation, it is assumed that there are six satellites evenly spaced in a circle with radius of 1000 m. The desired initial relative position and velocity of each satellite in local LVLH frame in order to form a circular formation are given by (5.70) (Inalhan et al. 2002; Pencil et al. 2004). The desired relative position and velocity of each satellite are then obtained by solving the relative dynamics in (2.83) with desired initial conditions.

$$\begin{cases} \vartheta_1 = 0^\circ, & \vartheta_2 = 60^\circ, & \vartheta_3 = 120^\circ, & \vartheta_4 = 180^\circ, & \vartheta_5 = 240^\circ, & \vartheta_6 = 300^\circ, \\ x_{i,0}^d = (R/2) \cos \vartheta_i, & x_{i,0}^{d'} = (R/2) \sin \vartheta_i, \\ y_{i,0}^d = 2x_{i,0}^{d'}, & y_{i,0}^{d'} = -(2+e)/(1+e)x_{i,0}^{d'}, \\ z_{i,0}^d = \sqrt{3}x_{i,0}, & z_{i,0}^{d'} = \sqrt{3}x_{i,0}^{d'}, & \forall i = 1, 2, \dots, 6 \end{cases} \quad (5.70)$$

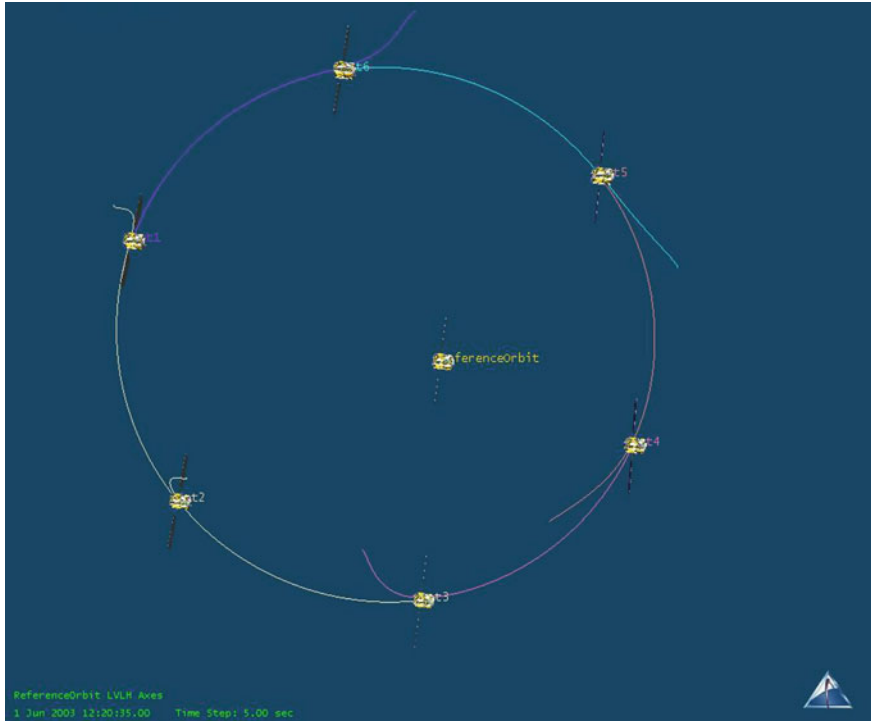


Fig. 5.12 Satellite motion under decentralized formation control and simulated with STK and MATLAB

The actual initial relative positions and velocities for the six satellites in the test scenario have initial errors as compared to the desired relative state, as given below:

$$\begin{aligned} \mathbf{x}_{1,0} &= 1.4\mathbf{x}_{1,0}^d, & \mathbf{x}_{2,0} &= 1.2\mathbf{x}_{2,0}^d, & \mathbf{x}_{3,0} &= 0.8\mathbf{x}_{3,0}^d, & \mathbf{x}_{4,0} &= 0.6\mathbf{x}_{4,0}^d, \\ \mathbf{x}_{5,0} &= 0.8\mathbf{x}_{5,0}^d, & \mathbf{x}_{6,0} &= 1.2\mathbf{x}_{6,0}^d \end{aligned} \quad (5.71)$$

The reference orbit is characterized by the following orbit elements:

$$a = 7350 \text{ km}, \quad e = 0.01, \quad i = 45^\circ, \quad \varpi = 30^\circ, \quad \Omega = 45^\circ, \quad \theta = 0^\circ \quad (5.72)$$

The minimum and maximum nonzero eigenvalues of L_G are $\lambda_{\min} = 1$, $\lambda_{\max} = 4$, respectively, since there are 6 satellites in the formation in the test scenario. The weight scalars for fuel balance among satellites is chosen as $\sigma_i = 1$. The choice of the state-feedback control gain, which is subject to the available maximum thrust, is a trade-off between fuel consumption and convergence rate. In order to save fuel consumption, quadratic performance weight matrices in (A.12) are chosen as $\bar{Q} = I_3$ and $\bar{R} = 10^{10}I_3$.

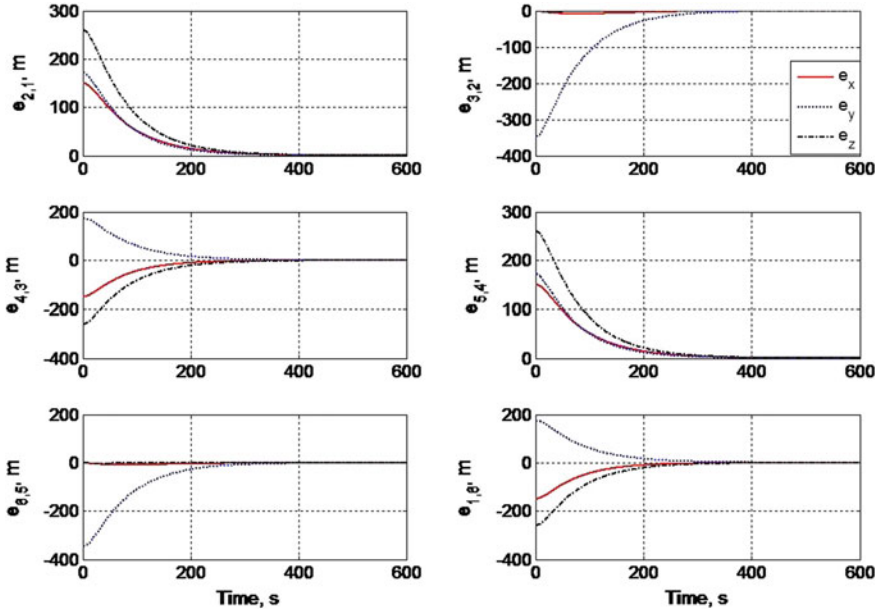


Fig. 5.13 Formation errors among satellites under decentralized formation control

The satellite motion under the decentralized formation control is shown in Fig. 5.12. As illustrated from this figure, the geometry center of the circular formation deviates from the reference orbit, because the relative state of each satellite with respect to other satellites rather than the one with respect to the reference orbit needs to be controlled. Figure 5.13 shows the formation error $e_{i,j} = \Delta \mathbf{x}_{i,j}^e$ between the i th and j th satellite. For ease of interpretation, the horizontal axis is described by time instead of true anomaly. As observed from this figure, the convergence time of formation control is 355 s assuming tolerant relative error is 1 %. Figure 5.14 shows the control acceleration of each satellite.

5.3.4.2 In-Plane Formation

For the desired line formation, it is assumed that there are five satellites are evenly spaced in the in-track direction with separation of 1000 m. The desired relative position and velocity of each satellite are obtained using the method in Pencil et al. (2004). The actual initial relative positions and velocities for the six satellites in the test scenario have initial errors as compared to the desired relative state. The satellite motion under the decentralized formation control is shown in Fig. 5.15, where the lines are maneuver trajectories. As can be seen from this figure, the desired in-plane formation is maintained from random positions.

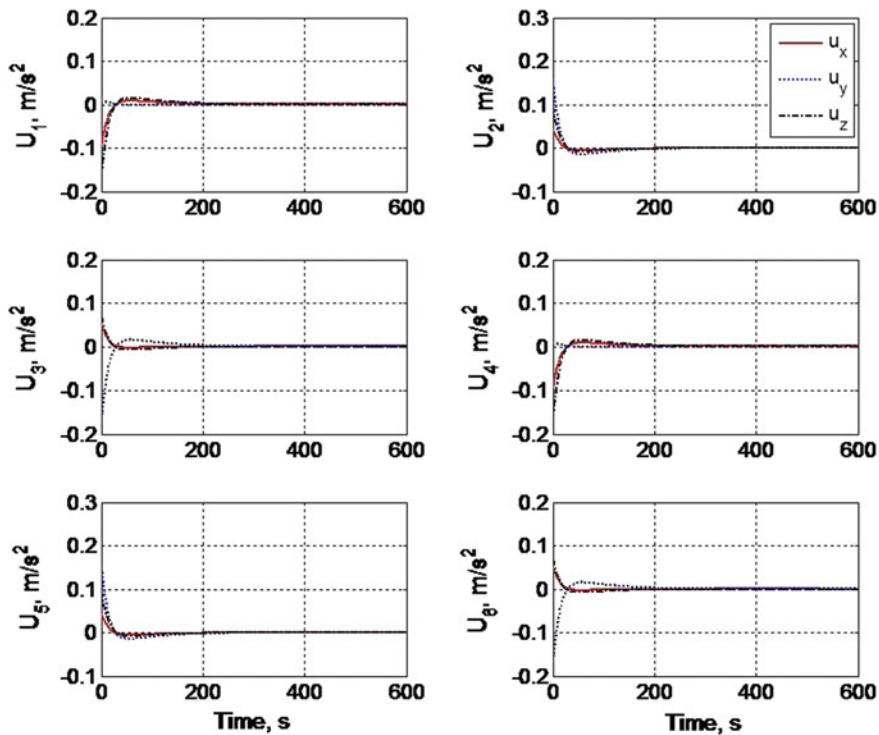


Fig. 5.14 Control acceleration of each satellite under decentralized formation control

5.4 Conclusions

In this chapter, two different kinds of continuous formation keeping approaches are proposed. Firstly, a real-time fuel-optimal control approach in the leader–follower frame is proposed to keep the formation against various orbital perturbations. Control saturation problem for low-thrust propulsion system is considered in the optimization problem. The resulting optimal control problem is then converted into a quadratic programming problem by application of Legendre pseudospectral method. To solve the smooth optimal control problem, Legendre pseudospectral method has two advantages over other numerical methods: improved accuracy and faster convergence rate which makes real-time optimal control feasible. Simulation results demonstrate the effectiveness of the proposed methodology for formation keeping problems.



Fig. 5.15 Satellite motion under decentralized formation control to maintain in-plane formation

Secondly, an efficient decentralized formation control method is presented in this chapter for satellite formation in mildly elliptic orbits using only local relative measurements. The desired formation is formulated as a general directed topology where the resulting graph has a spanning tree due to the availability of measurement between pairs of neighboring satellites. The goal of formation control for n satellites is achieved if the state-feedback controller stabilizes a system described by relative dynamics, differentiated by a scalar parameter. This scalar parameter takes values of the nonzero eigenvalues of a Laplacian matrix L representing the topology of relative measurements among the satellites. The proposed decentralized formation control architecture is completely decentralized and scalable. No infrastructure for full collaboration and communication among the formation satellites is required in the proposed scheme. Simulation results demonstrate the effectiveness of the proposed methodology for satellite formation control.

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Chapter 6

Decentralized Control for Attitude Synchronization Under Undirected Communication Topology

This chapter developed two decentralized coordinated attitude control laws for attitude synchronization and tracking problem with undirected (it is also named bidirectional) inter-satellite communication topology. Firstly, an adaptive sliding mode control law is developed against inertia matrix uncertainties and external disturbances. The proposed control law is based on the proposed multi-satellite sliding mode vector, on which each satellite approaches its desired time-varying attitude and angular velocity while achieving attitude synchronization among satellites in the formation. The first control law requires that each satellite knows its own angular velocity and the angular velocities of other satellites it can communicate with. However, angular velocity is not always available, because of either cost limitations or implementation considerations. As a remedy to such situation, a coordinated attitude control law without angular velocity measurements is then proposed. Passivity approach is deployed to remove the requirements for angular velocity measurements. The only assumption on communication topology for the proposed control laws is that the communication links are undirected. The stability of the resulting closed-loop systems is proved by virtue of Barbalat's Lemma. Simulation results demonstrate the effectiveness of the proposed attitude synchronization and tracking approaches.

6.1 Introduction

The need to maintain accurate relative orientation between spacecraft is critical in many satellite formation missions. For instance, in interferometry application, the relative orientation between spacecraft in a formation is required to be maintained precisely during formation maneuvers. In inter-spacecraft laser communication operation, the participating spacecraft are also required to maintain accurate relative attitude throughout the communication process. This control problem, commonly referred to as attitude synchronization in the literature, has attracted much research attention.

Various approaches have been proposed to investigate the problem of attitude synchronization. These solutions can be roughly categorized into four main approaches: leader-follower approach (Subbarao and Welsh 2008; Wang et al. 1999; Kang and Yeh 2002; Kristiansen et al. 2009), virtual structure approach (Beard et al. 2001; Ren and Beard 2004), behavior-based approach (Shan 2008; Chung et al. 2009; Lawton and Beard 2002; VanDyke and Hall 2006; Jin et al. 2008; Jin and Sun 2009), and graph-theoretical approach (Ren 2007a, b; Bai et al. 2008; Dimarogonas et al. 2009; Zou and Kumar 2012, 2013; Wu et al. 2014; Li and Kumar 2012; Zhou et al. 2013).

A leader-follower approach was deployed for attitude synchronization in Subbarao and Welsh (2008), Wang et al. (1999), Kang and Yeh (2002), Kristiansen et al. (2009), where the followers maintain the relative attitude with respect to the leader. But, as mentioned earlier, the leader-follower approach is a centralized control law, and the leader is a single point of failure for the formation. Centralized and decentralized implementations of a virtual structure coordination strategy were presented for attitude synchronization in Beard et al. (2001) and Ren and Beard (2004), respectively. In Shan (2008), Chung et al. (2009), Lawton and Beard (2002), VanDyke and Hall (2006), Jin et al. (2008), Jin and Sun (2009), a behavioral approach was used for attitude synchronization. In the behavior-based approach, each satellite has several basic behaviors, such as station-keeping and formation keeping. Each behavior generates a control input, and the control action for each satellite is a weighted average of control for each behavior. The behavioral approach lends itself naturally to a decentralized implementation. Graph-theoretical approach has been actively studied for cooperative control of multi-agent system using limited local interaction (Lawton et al. 2003; Fax and Murray 2004; Jadbabaie et al. 2003; Ren and Atkins 2007; Lafferriere et al. 2005; Khoo et al. 2009), and it has been investigated for attitude synchronization problem in Ren (2007a, b), Bai et al. (2008), Dimarogonas et al. (2009), Zou and Kumar 2012, 2013, Wu et al. (2014), Li and Kumar (2012) and Zhou et al. (2013).

The above-cited coordinated attitude control approaches were based on an assumption that each satellite knows its own angular velocity and the angular velocity of its neighbor. However, this assumption is not always satisfied, because of either cost limitations or implementation considerations. As a remedy to such situation, several researchers (Kristiansen et al. 2009; Lawton and Beard 2002; Ren 2009; Abdessameud and Tayebi 2009) proposed passivity-based attitude synchronization approaches without using angular velocity measurements. Lawton and Beard (2002) presented an attitude alignment law by use of passivity-based damping. However, the result in Lawton and Beard (2002) requires a bidirectional ring communication topology and relies on a restrictive domain of attraction for convergence analysis. Ren (2009) proposed a velocity-free distributed attitude synchronization approach by use of Modified Rodrigues Parameters for attitude representation. Minimal representation of attitude dynamics, such as Modified Rodrigues Parameters or Euler angles, contains singularities and is hence not suited for the design of globally stabilizing control algorithms. Kristiansen et al. (2009) presented a leader-follower approach for relative attitude tracking using feedback from attitude measurements

only. However, the leader-follower approach is a centralized controller, and the leader satellite is a single point of failure for the formation. Abdessameud and Tayebi (2009) proposed an attitude synchronization approach without velocity measurements using unit quaternion. The result in Abdessameud and Tayebi (2009) consists of associating an auxiliary nonlinear dynamic system to each satellite and to each pair of satellites with a communication link in order to recover and generate the necessary damping. Thus, this approach requires a lot of computation to solve auxiliary variables from those auxiliary nonlinear dynamic systems, especially when there are many communication links in a formation.

In the above decentralized attitude synchronization results, except Ren and Atkins (2007) and Ren (2010), it was assumed that the inter-satellite communication links are undirected (i.e., bidirectional). However, in practice, the inter-satellite communication topology may be restricted to be directed, such as in unidirectional satellite laser communication system. The control problem of attitude synchronization under directed communication topology is more challenging as compared to the case with undirected communication topology. This issue was studied in Ren and Atkins (2007) but the proposed control law requires derivative of the angular velocity, which may introduce additional noise into the system. Furthermore, the attitude tracking performance analysis in Ren and Atkins (2007) is applicable only to the case where the directed graph can be simplified to a graph with only one node. This constraint on communication topology is relaxed in Ren (2010), which uses Modified Rodrigues Parameters and Euler–Lagrange system to describe the satellite attitude dynamics. However, Modified Rodrigues Parameters contain singularity and are thus not suitable for the development of globally stabilizing control algorithms.

6.2 Satellite Attitude Dynamics

The rigid satellite attitude dynamics is described in this section. For the details, please refer to Wertz (1978). The unit quaternion, $\bar{\eta}$, represents the orientation of a body-fixed frame B with respect to an inertial frame I and is defined as

$$\bar{\eta} = \begin{bmatrix} \cos\left(\frac{\varphi}{2}\right) \\ \mathbf{e} \sin\left(\frac{\varphi}{2}\right) \end{bmatrix} = \begin{bmatrix} \eta_0 \\ \boldsymbol{\eta} \end{bmatrix} \quad (6.1)$$

where $\mathbf{e}$ is the Euler axis, φ is the Euler angle, $\boldsymbol{\eta} \in \mathbb{R}^3$, $\eta_0 \in \mathbb{R}$ are the vector part and scalar part of the quaternion $\bar{\eta}$, respectively, and satisfy the following constraint.

$$\boldsymbol{\eta}^T \boldsymbol{\eta} + \eta_0^2 = 1 \quad (6.2)$$

The product of two unit quaternion $\bar{\eta}_1$ and $\bar{\eta}_2$ is defined by

$$\bar{\eta}_1 \bar{\eta}_2 = \begin{bmatrix} \eta_{0,1}\eta_2 + \eta_{0,2}\eta_1 + \eta_1^\times \eta_2 \\ \eta_{0,1}\eta_{0,2} - \eta_1^T \eta_2 \end{bmatrix} \quad (6.3)$$

which is also a unit quaternion. The notation $a^\times$ for a vector $a = [a_1 \ a_2 \ a_3]^T$ is used to denote the skew-symmetric matrix.

$$a^\times = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \quad (6.4)$$

which satisfies the following important properties

$$\begin{aligned} a^\times b &= -b^\times a, \quad a^\times a = 0, \quad a^T b^\times a = 0 \\ a^\times b^\times &= ba^T - a^T b I_3, \quad (a^\times b)^\times = ba^T - ab^T \end{aligned} \quad (6.5)$$

The conjugate of the unit quaternion $\bar{\eta}$ is defined by $\bar{\eta}^* = [\eta_0, -\eta^T]^T$.

The dynamic and kinematic equations of a rigid satellite are described, respectively, as

$$J\dot{\omega} = -\omega^\times J\omega + u + d \quad (6.6)$$

and

$$\dot{\eta} = \frac{1}{2}(\eta^\times + \eta_0 I_3)\omega \quad (6.7)$$

$$\dot{\eta}_0 = -\frac{1}{2}\eta^T \omega \quad (6.8)$$

where $\omega \in \mathbb{R}^3$ denotes the angular velocity of satellite with respect to an inertial frame I and expressed in the body-fixed frame B ,

$$J = \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{12} & J_{22} & J_{23} \\ J_{13} & J_{23} & J_{33} \end{bmatrix} \quad (6.9)$$

is the constant, positive definite inertia matrix of the satellite and expressed in the body-fixed frame B , $u \in \mathbb{R}^3$ denotes the vector of control torque, $d \in \mathbb{R}^3$ denotes the vector of external disturbances, I_3 denotes a 3×3 identity matrix.

In the case of tracking a desired attitude motion, the attitude tracking problem is formulated similarly as in the related work (Ahmed et al. 1998; Luo et al. 2005). The desired motion of the satellite is specified by the attitude of the reference frame D . The target attitude of the satellite in the reference frame D with respect to the inertial frame I is described by unit quaternion $\bar{\eta}^d = [\eta_0^d, (\eta^d)^T]^T$ which satisfies $(\eta^d)^T \eta^d + (\eta_0^d)^2 = 1$. Let $\omega_d \in \mathbb{R}^3$ be the desired angular velocity of reference frame D with respect to the inertial frame I . The following assumptions are made about ω_d and $\dot{\omega}_d$.

Assumption 6.1 The desired angular velocity ω_d and its derivative $\dot{\omega}_d$ are bounded for all $t \geq 0$, i.e., there exists some finite constants $c_1 > 0$ and $c_2 > 0$ such that $\|\omega_d\| \leq c_1$ and $\|\dot{\omega}_d\| \leq c_2$ for all $t \geq 0$.

Let error quaternion $\bar{q} = [q_0, q^T]^T$ represent the orientation error of the body-fixed frame B with respect to the reference frame D and satisfies $q^T q + q_0^2 = 1$. $\bar{q}$ is related to $\bar{\eta}^d$ and $\bar{\eta}$ by quaternion multiplication $\bar{q} = (\bar{\eta}^d)^* \bar{\eta}$. The angular velocity error $\tilde{\omega}$ of frame B with respect to frame D is then represented by $\tilde{\omega} = \omega - R(\bar{q})\omega^d$, $R(\bar{q}) \in SO(3)$ is the rotation matrix from the reference frame D to the body-fixed frame B and is given by

$$R(\bar{q}) = (q_0^2 - q^T q)I_3 + 2qq^T - 2q_0 q^\times \quad (6.10)$$

where $SO(3)$ is the Lie group of orthogonal matrices with determinant 1. It follows from Wertz (1978) that $R^T R = 1$, $\|R\| = 1$, $\det(R) = 1$ and $\dot{R}(\bar{q}) = -\tilde{\omega}^\times R(\bar{q})$. Note that both $\bar{q}$ and $-\bar{q}$ stand for exactly the same physical attitude orientation, resulting in the same $R(\bar{q}) \in SO(3)$.

The rigid satellite attitude tracking error dynamics is described in Wertz (1978) and Ahmed et al. (1998).

$$J_i \dot{\tilde{\omega}}_i = -\omega_i^\times J_i \omega_i + J_i (\tilde{\omega}_i^\times R(\bar{q}_i) \omega^d - R(\bar{q}_i) \dot{\omega}^d) + u_i + d_i, \quad i = 1, \dots, n \quad (6.11)$$

$$\dot{q}_i = \frac{1}{2} (q_i^\times + q_{0i} I) \tilde{\omega}_i \triangleq \frac{1}{2} S(\bar{q}_i) \tilde{\omega}_i, \quad i = 1, \dots, n \quad (6.12)$$

$$\dot{q}_{0i} = -\frac{1}{2} q_i^T \tilde{\omega}_i, \quad i = 1, \dots, n \quad (6.13)$$

where superscript i denotes the i th satellite in a formation.

The attitude tracking problem of the i th satellite is solved if $\lim_{t \rightarrow \infty} q_i(t) \rightarrow 0$ and $\lim_{t \rightarrow \infty} \tilde{\omega}_i(t) \rightarrow 0$. It can be seen from (6.11) to (6.13) that the satellite attitude tracking problem is equivalent to an asymptotic stabilization problem for $\tilde{\omega}_i$ and q_i .

6.3 Problem Formulation of Attitude Synchronization

There are two objectives in attitude synchronization and tracking problem. The first objective is to rotate each individual satellite to its desired attitude and angular velocity. This behavior is called station-keeping behavior in VanDyke and Hall (2006). The second objective is to maintain the relative attitudes and relative angular velocities among satellites in the formation, called formation keeping behavior. The attitude synchronization and tracking problem is to find a continuous feedback control for each satellite in a formation by using the attitude information of itself and its neighboring satellites it can communicate with, so that the two objectives are achieved in the presence of inertia matrix uncertainties, external disturbances, and even inter-satellite communication delay.

The first objective is to rotate each individual satellite to its desired attitude and angular velocity as in the following equation. This behavior is called station-keeping behavior in VanDyke and Hall (2006).

$$\lim_{t \rightarrow \infty} \|\tilde{\omega}_i\| = \lim_{t \rightarrow \infty} \|q_i\| = 0, \quad i = 1, \dots, n \quad (6.14)$$

where $q_i, \tilde{\omega}_i$ are attitude tracking error and angular velocity error of the i th satellite, respectively.

The second objective is to maintain the relative attitudes and relative angular velocities among satellites in the formation, called formation keeping behavior, as in

$$\lim_{t \rightarrow \infty} \|\tilde{\omega}_i - \tilde{\omega}_j\| = \lim_{t \rightarrow \infty} \|q_i - q_j\| = 0, \quad i, j = 1, \dots, n, \quad i \neq j \quad (6.15)$$

Definition 6.1 The attitude synchronization and tracking problem is to find a continuous feedback control $u_i, i = 1, \dots, n$ for each satellite in a formation by using the attitude information of itself and its neighboring satellites it can communicate with, so that the two objectives in (6.14) and (6.15) are achieved.

The following assumption is made about the inter-satellite communication links in this chapter:

Assumption 6.2 The inter-satellite communication links are assumed to be undirected.

6.4 Decentralized Robust Adaptive Control for Attitude Synchronization

In this section, a decentralized adaptive sliding mode control law is proposed for attitude synchronization and tracking in the presence of inertia matrix uncertainties and external disturbances under a general undirected communication topology

(Wu et al. 2013). Firstly, the multi-satellite sliding mode vector is developed. Subsequently, a decentralized adaptive sliding mode control law is proposed. Finally, a smoothing control law is introduced to avoid the chattering of control input of the proposed control law.

The following assumptions are made about the dynamics of the attitude synchronization and tracking systems

Assumption 6.3 Let $J_i = \bar{J}_i + \Delta J_i$, where $\bar{J}_i, \Delta J_i$ are the nominal part and uncertain part of the inertia matrix of the i th satellite, respectively. The inertia matrix uncertainty ΔJ_i is assumed to satisfy $\|\Delta J_i\| \leq \gamma_{i,0}$;

Assumption 6.4 All the environmental disturbances due to gravitation, solar radiation pressure, and magnetic forces are assumed to be bounded, and aerodynamic drag is proportional to the square of angular velocity. Since the desired angular velocity ω_i^d is assumed to be bounded in Assumption 6.1, the external disturbances d_i are assumed to satisfy $\|d_i\| \leq \gamma_{i,1} + \gamma_{i,2}\|\tilde{\omega}_i\|^2$;

Assumption 6.5 The control law of each satellite might use angular velocity errors and error quaternions of its neighboring satellites in the cooperative attitude control problem, and error quaternion is bounded from its definition. Thus, the control torque u_i is assumed to satisfy $\|u_i\| \leq \xi_{i,0} + \xi_{i,1} \sum_{j \in N_i} \|\tilde{\omega}_j\| + \xi_{i,2} \sum_{j \in N_i} \|\tilde{\omega}_j\|^2$.

Where γ_{ij} and ξ_{ij} , $i = 1, \dots, n$, $j = 0, 1, 2$ are unknown non-negative constant, N_i denotes the i th satellite and all satellites which the i th satellite can communicate with, and $\|\cdot\|$ denotes the standard Euclidean vector norm or induced matrix norm as appropriate.

6.4.1 Multi-satellite Sliding Manifold

In this subsection, the multi-satellite sliding mode vector is developed in order to guarantee the two objectives of attitude synchronization and tracking in satellite formation and is defined as

$$S = [s_1, \dots, s_n]^T \quad (6.16)$$

where

$$s_i = b_i(\tilde{\omega}_i + Cq_i) + \sum_{j=1}^n a_{ij}[(\tilde{\omega}_i - \tilde{\omega}_j) + C(q_i - q_j)], \quad i = 1, \dots, n, j \neq i \quad (6.17)$$

with C is a positive definite constant matrix, scalar $b_i > 0$ is control gain for attitude tracking of the i th satellite (station-keeping behavior), scalar $a_{ij} \geq 0$ is control gain for inter-satellite attitude synchronization between the i th and j th satellite

(formation keeping behavior), and a_{ij} is the (i, j) th entry of the weighted adjacency matrix A associated with the undirected inter-satellite communication topology as defined in (A.3). A positive a_{ij} is referred to that a undirected communication link between the i th and the j th satellite is available, and the behavior weights are restricted to be $a_{ij} = a_{ji}$.

Remark 6.1 In the multi-satellite sliding mode vector defined in (6.16) and (6.17), in addition to attitude error and angular velocity error of each individual satellite, the relative attitude errors and relative angular velocity errors among satellites are also included. Hence, under the sliding mode control proposed in the next section, each satellite approaches its desired attitude and angular velocity while achieving attitude synchronization among satellites in the formation. This gives rise to the concept of attitude synchronization and tracking.

Using the Kronecker product and the algebraic graph theory in Appendix A, the multi-satellite sliding vector in (6.16) can be rewritten as

$$S = [(L + B) \otimes I_3](\tilde{\Omega} + \tilde{C}Q) \quad (6.18)$$

where $\tilde{\Omega} = [\tilde{\omega}_1, \dots, \tilde{\omega}_n]^T$, $B = \text{diag}[b_1, \dots, b_n]$, $\tilde{C} = \text{diag}[C, \dots, C] \in \mathbb{R}^{3n \times 3n}$, I_3 denotes the 3×3 identity matrix, $Q = [q_1, \dots, q_n]^T$, L is the weighted Laplacian matrix corresponding to the inter-satellite undirected communication topology, control gain a_{ij} for inter-satellite attitude synchronization are the weights in L .

The multi-satellite sliding mode surface is then defined as $S = 0$ or

$$[(L + B) \otimes I_3](\tilde{\Omega} + \tilde{C}Q) = 0 \quad (6.19)$$

The following theorem defines the condition under which the attitude synchronization and tracking will be achieved.

Theorem 6.1 *Consider satellite formation attitude tracking systems described by (6.11)–(6.13), if Assumption 6.2 is valid, then on the multi-satellite sliding mode surface $S = 0$, the attitude synchronization and tracking problem in Definition 6.1 is asymptotically solved.*

Proof Since the inter-satellite communication links are undirected, all eigenvalues of the weighted Laplacian matrix L are non-negative through Proposition A.3. Moreover, B is a diagonal matrix with all positive elements in the diagonal. Thus, all eigenvalues of $L + B$ are positive. With Lemma 5.1, all eigenvalues of $(L + B) \otimes I_3$ are positive. Furthermore, L is symmetrical, and so is $(L + B) \otimes I_3$ from Lemma 5.1. Consequently, it follows that $(L + B) \otimes I_3$ is positive definite and non-singular.

Since $(L + B) \otimes I_3$ is non-singular, on the sliding mode surface $S = 0$, (6.19) yields

$$\tilde{\Omega} + \tilde{C}Q = 0 \quad (6.20)$$

which is equivalent to

$$\tilde{\omega}_i + Cq_i = 0, \quad i = 1, \dots, n \quad (6.21)$$

As proved in Li and Wang (2007), the above equation implies that

$$\lim_{t \rightarrow \infty} \|\tilde{\omega}_i\| = \lim_{t \rightarrow \infty} \|q_i\| = 0, \quad i = 1, \dots, n \quad (6.22)$$

It can be concluded that on the multi-satellite sliding mode surface $S = 0$, the attitude synchronization and tracking problem in Definition 6.1 is asymptotically solved. $\square$

6.4.2 Decentralized Adaptive Sliding Mode Control Design

This subsection designs the equivalent control-based sliding mode control laws such that the reaching and sliding conditions are satisfied.

To develop the control law, the following equations are derived from (6.11) and (6.12)

$$\dot{\tilde{\omega}}_i + C\dot{q}_i = h_i + \rho_i + \bar{J}_i^{-1}u_i, \quad i = 1, \dots, n \quad (6.23)$$

with

$$h_i(t) \triangleq -\bar{J}_i^{-1}\omega_i^\times \bar{J}_i \omega_i + (\tilde{\omega}_i^\times R(\bar{q}_i)\omega_i^d - R(\bar{q}_i)\dot{\omega}_i^d) + \frac{1}{2}C(q_i^\times + q_{i,0}I)\tilde{\omega}_i \quad (6.24)$$

$$\rho_i(t) \triangleq \bar{J}_i^{-1}[d_i - \Delta J_i \dot{\tilde{\omega}}_i - \omega_i^\times \Delta J_i \omega_i + \Delta J_i(\tilde{\omega}_i^\times R(\bar{q}_i)\omega_i^d - R(\bar{q}_i)\dot{\omega}_i^d)] \quad (6.25)$$

$\rho_i(t)$ represents the inertia matrix uncertainties and external disturbances. Under Assumptions 6.1 and 6.3–6.5, it can be verified that $\rho_i(t)$ is bounded by the following function

$$\|\rho_i(t)\|_1 \leq d_{i,0} + d_{i,1} \sum_{j \in N_i} \|\tilde{\omega}_j\|_1 + d_{i,2} \sum_{j \in N_i} \|\tilde{\omega}_j\|_1^2, \quad i = 1, \dots, n \quad (6.26)$$

where $d_{i,0}$, $d_{i,1}$, $d_{i,2}$ are positive constant numbers, and $\|\cdot\|_1$ denotes the induced matrix 1-norm.

Sliding mode control is deployed in this section for attitude synchronization and tracking in the presence of model uncertainties and external disturbances. However, in those conventional sliding mode control laws in Lo and Chen (1995), Crassidis et al. (2000) and Boškovic et al. (2001), an important assumption is that inertia uncertainties and external disturbances are bounded and that their bounds are available to the designer. These bounds are an important clue to guarantee the

stability of the closed-loop system. However, bounds on the uncertainties of satellite are not easily obtained because of the complexity of the structure of the uncertainties. In particular, the magnitude of external disturbances cannot be easily estimated, since various environmental disturbances exist, such as gravity-gradient torque, aerodynamic torque, earth magnetic torque, and solar radiation torque. In many practical situations, even if the bounds can be learned, they may be conservative. The implementation of the control law, based on these conservative bounds, may result in impractically large control authority and control chattering. Control chattering is highly undesirable in satellite mission, since it implies extremely high control activity and may excite neglected high-frequency dynamics. Therefore, a simple methodology is required to acquire the bounds on uncertainties. Adaptive approach introduced in the following offers a simple and effective tool to this problem.

Suppose $\hat{d}_{i,0}$, $\hat{d}_{i,1}$, and $\hat{d}_{i,2}$ are estimations of $d_{i,0}$, $d_{i,1}$, and $d_{i,2}$, respectively. Then, the estimation errors are defined as $\tilde{d}_{i,0} = \hat{d}_{i,0} - d_{i,0}$, $\tilde{d}_{i,1} = \hat{d}_{i,1} - d_{i,1}$ and $\tilde{d}_{i,2} = \hat{d}_{i,2} - d_{i,2}$.

Now, consider the simple adaptation laws for the upper bound of the norm $\|\rho_i(t)\|_1$ such that

$$\begin{aligned}\dot{\hat{d}}_{i,0} &\triangleq \kappa_{i,0} \|s_i\|_1 \\ \dot{\hat{d}}_{i,1} &\triangleq \kappa_{i,1} \|s_i\|_1 \sum_{j \in N_i} \|\tilde{\omega}_j\|_1 \\ \dot{\hat{d}}_{i,2} &\triangleq \kappa_{i,2} \|s_i\|_1 \sum_{j \in N_i} \|\tilde{\omega}_j\|_1^2\end{aligned}\tag{6.27}$$

where $\kappa_{i,0}$, $\kappa_{i,1}$ and $\kappa_{i,2}$ are positive adaptive gains.

Since $d_{i,0}$, $d_{i,1}$, and $d_{i,2}$ are assumed to be constant, $\dot{\hat{d}}_{i,0} = \dot{\tilde{d}}_{i,0}$, $\dot{\hat{d}}_{i,1} = \dot{\tilde{d}}_{i,1}$, $\dot{\hat{d}}_{i,2} = \dot{\tilde{d}}_{i,2}$. Then, the adaptive parameters can be obtained by integrating

$$\begin{aligned}\hat{d}_{i,0}(t) &\triangleq \hat{d}_{i,0}^0 + \kappa_{i,0} \int_{t_0}^t \|s_i\|_1 dt \\ \hat{d}_{i,1}(t) &\triangleq \hat{d}_{i,1}^0 + \kappa_{i,1} \int_{t_0}^t \|s_i\|_1 \sum_{j \in N_i} \|\tilde{\omega}_j\|_1 dt \\ \hat{d}_{i,2}(t) &\triangleq \hat{d}_{i,2}^0 + \kappa_{i,2} \int_{t_0}^t \|s_i\|_1 \sum_{j \in N_i} \|\tilde{\omega}_j\|_1^2 dt\end{aligned}\tag{6.28}$$

where $\hat{d}_{i,0}^0$, $\hat{d}_{i,1}^0$ and $\hat{d}_{i,2}^0$ are the initial values of $\hat{d}_{i,0}$, $\hat{d}_{i,1}$ and $\hat{d}_{i,2}$. By choosing appropriate $\{\hat{d}_{i,0}^0, \hat{d}_{i,1}^0, \hat{d}_{i,2}^0\}$ and $\{\kappa_{i,0}, \kappa_{i,1}, \kappa_{i,2}\}$, the rate of parameter adaptation can be adjusted.

The control input for the i th satellite is proposed as

$$u_i = -\bar{J}_i[h_i + K_i s_i + \hat{\alpha}_i \operatorname{sgn}(s_i)], \quad i = 1, \dots, n \quad (6.29)$$

where $\bar{J}_i$ is the nominal inertia matrix of the i th satellite, K_i is the positive definite gain matrix, h_i is defined in (6.24), s_i is the component of multi-satellite sliding mode vector S defined in (6.17), and $\operatorname{sgn}(s_i) \triangleq [\operatorname{sgn}(s_{i,1}) \quad \operatorname{sgn}(s_{i,2}) \quad \operatorname{sgn}(s_{i,3})]^T$, $s_{i,j}$, $j = 1, 2, 3$ is the j th component of s_i , $\operatorname{sgn}(\cdot)$ denotes the sign function, i.e.,

$$\operatorname{sgn}(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases} \quad (6.30)$$

and $\hat{\alpha}_i$ is the adaptive upper bound of the norm $\|\rho_i(t)\|_1$ and is defined by

$$\hat{\alpha}_i \triangleq \hat{d}_{i,0} + \hat{d}_{i,1} \sum_{j \in N_i} \|\tilde{\omega}_j\|_1 + \hat{d}_{i,2} \sum_{j \in N_i} \|\tilde{\omega}_j\|_1^2 \quad (6.31)$$

The following theorem gives the condition for the existence of the multi-satellite sliding mode surface (6.19) for satellite formation attitude tracking systems described by (6.11)–(6.13).

Theorem 6.2 *Consider satellite formation attitude tracking systems described by (6.11)–(6.13) with the decentralized adaptive sliding mode control law (6.29) and the parameter adaptation law (6.27). If Assumptions 6.1–6.5 are valid, then the multi-satellite sliding mode surface $S = 0$ defined in (6.16) and (6.17) will be reached asymptotically.*

Proof The candidate of Lyapunov function is set as

$$V = V_1 + V_2 \quad (6.32)$$

with

$$V_1 = \frac{1}{2} S^T [(L+B) \otimes I_3]^{-1} S \quad (6.33)$$

$$V_2 = \frac{1}{2} \sum_{i=1}^n \left(\kappa_{i,0}^{-1} \tilde{d}_{i,0}^2 + \kappa_{i,1}^{-1} \tilde{d}_{i,1}^2 + \kappa_{i,2}^{-1} \tilde{d}_{i,2}^2 \right) \quad (6.34)$$

As proved in previous subsection, $(L+B) \otimes I_3$ is positive definite, so $[(L+B) \otimes I_3]^{-1}$ is positive definite. Thus, V is positive definite and radially unbounded. $\square$

Taking the first derivative of V_1 and adopting (6.18) yields

$$\begin{aligned}
 \dot{V}_1 &= S^T [(L+B) \otimes I_3]^{-1} \dot{S} \\
 &= S^T [(L+B) \otimes I_3]^{-1} [(L+B) \otimes I_3] \begin{bmatrix} \dot{\omega}_1 + C\dot{q}_1 \\ \vdots \\ \dot{\omega}_n + C\dot{q}_n \end{bmatrix} \\
 &= \sum_{i=1}^n s_i^T (\dot{\omega}_i + C\dot{q}_i)
 \end{aligned} \tag{6.35}$$

Substituting (6.23) into the above equation leads to

$$\dot{V}_1 = \sum_{i=1}^n s_i^T (h_i + \rho_i + \bar{J}_i^{-1} u_i) \tag{6.36}$$

Substituting the control input (6.29) into the above equation and adopting (6.26) and (6.31) yields

$$\begin{aligned}
 \dot{V}_1 &= \sum_{i=1}^n [-s_i^T K_i s_i - s_i^T (\bar{\alpha}_i \operatorname{sgn}(s_i)) + s_i^T \rho_i] \\
 &= \sum_{i=1}^n (-s_i^T K_i s_i - \hat{\alpha}_i \|s_i\|_1 + s_i^T \rho_i) \\
 &\leq \sum_{i=1}^n (-s_i^T K_i s_i - \hat{\alpha}_i \|s_i\|_1 + \|s_i\|_1 \cdot \|\rho_i\|_1) \\
 &\leq \sum_{i=1}^n \left[-s_i^T K_i s_i - \left(\hat{d}_{i,0} + \hat{d}_{i,1} \sum_{j \in N_i} \|\tilde{\omega}_j\|_1 + \hat{d}_{i,2} \sum_{j \in N_i} \|\tilde{\omega}_j\|_1^2 \right) \|s\|_1 \right] \\
 &\quad + \sum_{i=1}^n \left[\|s_i\|_1 \cdot \left(d_{i,0} + d_{i,1} \sum_{j \in N_i} \|\tilde{\omega}_j\|_1 + d_{i,2} \sum_{j \in N_i} \|\tilde{\omega}_j\|_1^2 \right) \right] \\
 &= \sum_{i=1}^n \left[-s_i^T K_i s_i - \left(\tilde{d}_{i,0} + \tilde{d}_{i,1} \sum_{j \in N_i} \|\tilde{\omega}_j\|_1 + \tilde{d}_{i,2} \sum_{j \in N_i} \|\tilde{\omega}_j\|_1^2 \right) \|s_i\|_1 \right]
 \end{aligned} \tag{6.37}$$

Taking the first derivative of V_2 and adopting (6.27) yields

$$\begin{aligned}
 \dot{V}_2 &= \sum_{i=1}^n \left(\kappa_{i,0}^{-1} \tilde{d}_{i,0} \dot{\tilde{d}}_{i,0} + \kappa_{i,1}^{-1} \tilde{d}_{i,1} \dot{\tilde{d}}_{i,1} + \kappa_{i,2}^{-1} \tilde{d}_{i,2} \dot{\tilde{d}}_{i,2} \right) \\
 &= \sum_{i=1}^n \left(\tilde{d}_{i,0} + \tilde{d}_{i,1} \sum_{j \in N_i} \|\tilde{\omega}_j\|_1 + \tilde{d}_{i,2} \sum_{j \in N_i} \|\tilde{\omega}_j\|_1^2 \right) \|s_i\|_1
 \end{aligned} \tag{6.38}$$

Combining the above equation with (6.37) leads to

$$\dot{V} \leq - \sum_{i=1}^n s_i^T K_i s_i \quad (6.39)$$

Therefore, it follows that $s_i \in L^\infty$ and $\tilde{d}_{i,0}, \tilde{d}_{i,1}, \tilde{d}_{i,2} \in L^\infty$. Consequently, $u_i \in L^\infty$ with (6.29) and Assumption 6.5. It follows that $\dot{\tilde{w}}_i, \dot{q}_i$ and hence $\dot{s}_i$ are all bounded from (6.11) and (6.12). Integrating $\dot{V}$ gives the results that $s_i \in L^2$. Hence, using the corollary of Barbalat's Lemma in Sastry and Bodson (1989), it follows that $\lim_{t \rightarrow \infty} s_i(t) = 0, i = 1, \dots, n$. Thus, it is obtained that $\lim_{t \rightarrow \infty} S(t) = 0$.

Remark 6.2 There is no extra restriction on parameters $b_i > 0, a_{ij} \geq 0$ in (6.17) unlike VanDyke and Hall (2006), Jin et al. (2008) and Ren (2007a). So the proposed attitude synchronization and tracking scheme will enable the designer to prioritize between station-keeping behavior and formation keeping behavior. For example, if one wants to prioritize formation keeping behavior, a_{ij} should be increased while b_i should be decreased or invariant.

Remark 6.3 In the proposed control law, the desired attitude of each satellite with respect to inertial frame I is not restricted to be same. Thus, the given relative attitude among satellites can be maintained. Note that $\bar{q}_{i,j}^d = (\bar{q}_j^d)^T \bar{q}_i^d$ defines the relative attitude of the i th with respect to the j th satellite, where $\bar{q}_i^d$ and $\bar{q}_j^d$ denote the relative attitude of the i th and j th satellite with respect to inertial frame I , respectively.

Remark 6.4 Reference Jin et al. (2008) presents a sliding mode controller for attitude synchronization and stabilization. Unlike Jin et al. (2008), the proposed control law is for attitude synchronization and tracking and is based on graph theory and the proposed multi-satellite sliding mode vector.

6.4.3 Smoothing Control Law

The proposed control law in previous subsection is discontinuous across the surface $S(t)$, thus leads to control chattering. This situation can be remedied by smoothing out the control discontinuity in a thin boundary layer neighboring the switching surface. To do this, the sign function in the control law (6.29) can be replaced by a saturation function, which is defined as

$$\text{sat}(x) = \begin{cases} 1 & \text{if } \frac{x}{\phi} \geq 1 \\ \frac{x}{\phi} & \text{if } -1 < \frac{x}{\phi} < 1 \\ -1 & \text{if } \frac{x}{\phi} \leq -1 \end{cases} \quad (6.40)$$

where ϕ is the boundary layer thickness. As ϕ tends to be zero, the performance of control law with this boundary layer can be made arbitrarily close to that of original control law (6.29). The practical advantages of control law with this boundary layer may be significant, although it leads to small terminal tracking error. However, it should be pointed out that in this case, the estimated gains $\tilde{d}_0, \tilde{d}_1$ and $\tilde{d}_2$ may become unbounded in the boundary layer since the restriction to the sliding surface cannot always be achieved.

In order to address this problem, the adaptive sliding mode control law given in (6.29) is modified as

$$u_i = -\bar{J}_i[h_i + K_i \bar{s}_i + \hat{\alpha}_i \text{sat}(s_i)], \quad i = 1, \dots, n \quad (6.41)$$

where $\text{sat}(s_i) \triangleq [\text{sat}(s_{i,1}) \quad \text{sat}(s_{i,2}) \quad \text{sat}(s_{i,3})]^T$, $\bar{s}_i = [\bar{s}_{i,1}, \bar{s}_{i,2}, \bar{s}_{i,3}]^T$ is a measure of the algebraic distance of the current state to the boundary layer (Slotine and Coetsee 1986), $\bar{s}_{i,j} = s_{i,j} - \phi_i \text{sat}(s_{i,j})$, $i = 1, \dots, n, j = 1, 2, 3$.

The adaptation laws (6.27) are modified as

$$\begin{aligned} \dot{\tilde{d}}_{i,0} &\triangleq \kappa_{i,0} \|\bar{s}_i\|_1 \\ \dot{\tilde{d}}_{i,1} &\triangleq \kappa_{i,1} \|\bar{s}_i\|_1 \sum_{j \in N_i} \|\tilde{\omega}_j\|_1 \\ \dot{\tilde{d}}_{i,2} &\triangleq \kappa_{i,2} \|\bar{s}_i\|_1 \sum_{j \in N_i} \|\tilde{\omega}_j\|_1^2 \end{aligned} \quad (6.42)$$

The following Lyapunov function is defined in order to prove the convergence to the boundary layer:

$$\bar{V} = \frac{1}{2} \bar{S}^T [(L+B) \otimes I_3]^{-1} \bar{S} + \frac{1}{2} \sum_{i=1}^n \left(\kappa_{i,0}^{-1} \tilde{d}_{i,0}^2 + \kappa_{i,1}^{-1} \tilde{d}_{i,1}^2 + \kappa_{i,2}^{-1} \tilde{d}_{i,2}^2 \right) \quad (6.43)$$

where $\bar{S} = [\bar{s}_1, \dots, \bar{s}_n]$.

Noting that $\dot{\bar{S}} = \dot{S}$ outside the boundary layer, while $\bar{S} = 0$ inside the boundary layer leads to

$$\dot{\bar{V}} \leq - \sum_{i=1}^n (\bar{s}_i)^T K_i \bar{s}_i \quad (6.44)$$

so that $\dot{\bar{V}} \leq 0$ outside the boundary layer. Finally, (6.43) implies that $\dot{\bar{V}} = 0$ inside the boundary layer, which shows that $\dot{\bar{V}} \leq 0$ is valid everywhere and thus guarantees further that states eventually converge to the boundary layer.

It can be shown that the system is globally uniformly ultimately bounded inside the boundary layer (Slotine and Coetsee 1986).

Remark 6.5 As seen from (6.42) that adaptation ceased as soon as the boundary layer is reached. This avoids the undesirable long-term drift found in many adaptive schemes and provides a consistent rule on when to stop adaptation (Slotine and Coetsee 1986).

6.5 Velocity-Free Coordinated Attitude Control

This section generalizes the passivity approach in Kristiansen et al. (2009), Lizarralde and Wen (1996) and Tsiotras (1998) for attitude synchronization in satellite formation, where angular velocity measurements are not required (Wu et al. 2010).

Assumption 6.6 For simplicity, it is assumed that the desired orientations and angular velocities of all satellites are identical, i.e.,

$$\bar{\eta}_1^d(t) = \cdots = \bar{\eta}_n^d(t) = \bar{\eta}^d(t), \quad \omega_1^d(t) = \cdots = \omega_n^d(t) = \omega^d(t) \quad (6.45)$$

Furthermore, each satellite knows the desired orientation and angular velocity.

For simplicity, the attitude tracking error kinematic equations in (6.12) and (6.13) are combined as follows:

$$\dot{\bar{q}}_i = \frac{1}{2} A(\bar{q}_i) \tilde{\omega}_i \quad i = 1, \dots, n \quad (6.46)$$

with

$$A(\bar{q}_i) = \begin{bmatrix} -q_i^T \\ S(\bar{q}_i) \end{bmatrix} \quad (6.47)$$

The following property about $A(\bar{q}_i)$ is useful.

$$A(\bar{q}_i)^T A(\bar{q}_i) = I_3 \quad (6.48)$$

The dynamic control action for the i th satellite in a formation is proposed as follows:

$$\dot{x}_i = Ax_i + k_i q_i + \sum_{j=1}^n k_{ij} (q_i - q_j) \quad (6.49)$$

$$y_i = PAx_i + k_i Pq_i + P \sum_{j=1}^n k_{ij} (q_i - q_j) \quad (6.50)$$

$$u_i = -\rho_i q_i - \sum_{j=1}^n \rho_{ij} q_{ij} - S^T(\bar{q}_i) y_i + (R(\bar{q}_i) \omega^d)^\times J_i (R(\bar{q}_i) \omega^d) + J_i R(\bar{q}_i) \dot{\omega}^d \quad (6.51)$$

where $i = 1, \dots, n$, $A \in \mathbb{R}^{3 \times 3}$ is Hurwitz, $P \in \mathbb{R}^{3 \times 3}$ is a positive definite matrix, which is the solution of the Lyapunov equation $A^T P + PA = -Q$ with $Q \in \mathbb{R}^{3 \times 3}$ being a positive definite matrix, k_i is a positive scalar, $\rho_i > 0$ is control gain for attitude tracking of each individual satellite, ρ_{ij} is control gain for attitude synchronization among satellites, If inter-satellite undirected communication between the i th and the j th satellite is available, then it is assumed that $\rho_{ij} = \rho_{ji} > 0$ and $k_{ij} = k_{ji} > 0$, otherwise $\rho_{ij} = \rho_{ji} = 0$ and $k_{ij} = k_{ji} = 0$.

q_{ij} is the vector part of $\bar{q}_{ij}$. From Assumption 6.5, it can be verified that $\bar{q}_{ij}$ is equal to the relative quaternions $\bar{\eta}_{ij}$ as follows

$$\bar{q}_{ij} = (\bar{q}_j)^* \bar{q}_i = [(\bar{\eta}^d)^* \bar{\eta}_j]^* [(\bar{\eta}^d)^* \bar{\eta}_i] = (\bar{\eta}_j)^* \bar{\eta}^d (\bar{\eta}^d)^* \bar{\eta}_i = (\bar{\eta}_j)^* \bar{\eta}_i = \bar{\eta}_{ij} \quad (6.52)$$

$\bar{\eta}_{ij}$ representing the relative attitude of the i th with respect to j th satellite, which can be obtained by the inter-satellite communication or relative sensing.

Remark 6.6 The proposed controller (6.51) has a clear intuitive structure. The term $-\rho_i q_i$ is for absolute attitude tracking for each individual satellite, and the term $\sum_{j=1}^n \rho_{ij} q_{ij}$ is for attitude synchronization among satellites. The term $-E^T(\bar{q}_i) y_i$ is to recover and generate the necessary damping that would have been generated by the angular velocities and the relative angular velocities. The term $(R(\bar{q}_i) \omega^d)^\times J_i (R(\bar{q}_i) \omega^d) + J_i R(\bar{q}_i) \dot{\omega}^d$ is feedforward compensation for absolute attitude tracking. Equations (6.49) and (6.50) take the role of a filter whose output y_i provides damping to the formation. The term $k_i q_i$ in (6.49) is used to introduce absolute damping for each individual satellite, and the term $\sum_{j=1}^n k_{ij} (q_i - q_j)$ in (6.49) is used to generate relative damping between neighboring satellites.

The stability of the proposed control law is given in the following theorem.

Theorem 6.3 Consider satellite attitude tracking systems described by (6.11)–(6.13) with the decentralized coordinated attitude control law (6.49)–(6.51), if the Assumptions 6.2 and 6.6 are valid and the control gains satisfy

$$\rho_i > 2 \sum_{j=1}^n \rho_{ij}, \quad i = 1, \dots, n \quad (6.53)$$

then the attitude synchronization and tracking problem in Definition 6.1 is asymptotically solved. Furthermore, if there exist a time $T > 0$ such that $q_{0,i}(t) > 0$ for all $t \geq T$ and $i = 1, \dots, n$, then the same convergence results are obtained without condition (6.53).

Proof Choose the candidate of Lyapunov function as

$$V = \frac{1}{2} \sum_{i=1}^n \tilde{\omega}_i^T J_i \tilde{\omega}_i + \sum_{i=1}^n \rho_i \left[(1 - q_{0,i})^2 + q_i^T q_i \right] + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \rho_{ij} \left[(1 - q_{0,ij})^2 + q_{ij}^T q_{ij} \right] + V_1 \quad (6.54)$$

with

$$V_1 = \frac{1}{2} \dot{x}^T (M \otimes I_3)^{-1} (I_n \otimes P) \dot{x} \quad (6.55)$$

where $\otimes$ denotes Kronecker product, $\dot{x}$ is column stack vector of all $\dot{x}_i, i = 1, \dots, n, M \triangleq L + \text{diag}[k_1, \dots, k_n]$, with $L = [l_{ij}] \in \mathbb{R}^{n \times n}$ defined as $l_{ij} = -k_{ij}$ and $l_{ii} = \sum_{j \neq i} k_{ij}$. M is symmetric positive definite from algebraic graph theorem in Appendix A, thus $(M \otimes I_3)^{-1}$ is also symmetric positive definite by use of the property of Kronecker product in Lemma 5.1. Note that $I_n \otimes P$ is symmetric positive definite. It is easy to verify that $(M \otimes I_3)^{-1}$ and $(I_n \otimes P)$ commute. Thus, it follows that $(M \otimes I_3)^{-1} (I_n \otimes P)$ is also symmetric positive definite. Therefore, the function V_1 is positive definite and radially unbounded, so is V . $\square$

From (6.13), the following equation is obtained

$$\dot{q}_{0,ij} = -\frac{1}{2} (\tilde{\omega}_{ij})^T q_{ij}, \quad i, j = 1, n, i \neq j \quad (6.56)$$

Taking the first derivative of V and adopting (6.8) and the above equation yields

$$\dot{V} = \sum_{i=1}^n \tilde{\omega}_i^T J_i \dot{\tilde{\omega}}_i + \sum_{i=1}^n \rho_i \tilde{\omega}_i^T q_i + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \rho_{ij} \tilde{\omega}_{ij}^T q_{ij} + \dot{V}_1 \quad (6.57)$$

Note that

$$\begin{aligned} \tilde{\omega}_i^T J_i \dot{\tilde{\omega}}_i &= -\tilde{\omega}_i^T \left[(R(\bar{q}_i) \omega^d)^\times J_i + J_i (R(\bar{q}_i) \omega^d)^\times \right] \tilde{\omega}_i \\ &\quad - \tilde{\omega}_i^T \left[(R(\bar{q}_i) \omega^d)^\times J_i (R(\bar{q}_i) \omega^d) + J_i R(\bar{q}_i) \dot{\omega}^d \right] + \tilde{\omega}_i^T u_i \\ &= -\tilde{\omega}_i^T \left[(R(\bar{q}_i) \omega^d)^\times J_i (R(\bar{q}_i) \omega^d) + J_i R(\bar{q}_i) \dot{\omega}^d \right] + \tilde{\omega}_i^T u_i \end{aligned} \quad (6.58)$$

where $\tilde{\omega}_i^T \left[(R(\bar{q}_i) \omega^d)^\times J_i + J_i (R(\bar{q}_i) \omega^d)^\times \right] \tilde{\omega}_i = 0$ is used to obtain the last equality. It can be readily shown that $(R(\bar{q}_i) \omega^d)^\times J_i + J_i (R(\bar{q}_i) \omega^d)^\times$ is skew-symmetric due to the fact that J_i is symmetric and $R(\bar{q}_i) \omega^d$ is skew-symmetric. Substituting the control inputs (6.51) and (6.58) into (6.57) leads to

$$\dot{V} = - \sum_{i=1}^n \tilde{\omega}_i^T \sum_{j=1}^n \rho_{ij} q_{ij} - \sum_{i=1}^n \tilde{\omega}_i^T S^T(\bar{q}_i) y_i + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \rho_{ij} \tilde{\omega}_{ij}^T q_{ij} + \dot{V}_1 \quad (6.59)$$

Note that

$$\begin{aligned} & - \sum_{i=1}^n \tilde{\omega}_i^T \sum_{j=1}^n \rho_{ij} q_{ij} + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \rho_{ij} (\tilde{\omega}_{ij})^T q_{ij} \\ &= - \frac{1}{2} \sum_{i=1}^n \tilde{\omega}_i^T \sum_{j=1}^n \rho_{ij} q_{ij} - \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \rho_{ij} \tilde{\omega}_j^T q_{ji} + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \rho_{ij} (\tilde{\omega}_{ij})^T q_{ij} \\ &= - \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \left[\rho_{ij} \tilde{\omega}_i^T q_{ij} + \rho_{ij} \tilde{\omega}_j^T (-R(\bar{q}_{ji}) q_{ij}) \right] + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \rho_{ij} (\tilde{\omega}_{ij})^T q_{ij} \quad (6.60) \\ &= - \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \left[\rho_{ij} (\tilde{\omega}_i - R(\bar{q}_{ij}) \tilde{\omega}_j)^T q_{ij} \right] + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \rho_{ij} (\tilde{\omega}_{ij})^T q_{ij} \\ &= 0 \end{aligned}$$

where the fact $\rho_{ij} = \rho_{ji}$ is used to obtain the second equality.

Equation (6.49) can be rewritten as follows

$$\dot{x} = (I_n \otimes A)x + (M \otimes I_3)q \quad (6.61)$$

where q is column stack vector of all $q_i, i = 1, \dots, n$.

Differentiating both sides of the above equation yields

$$\ddot{x} = (I_n \otimes A)\dot{x} + (M \otimes I_3)\dot{q} \quad (6.62)$$

Taking the first derivative of V_1 leads to

$$\begin{aligned} \dot{V}_1 &= \frac{1}{2} \dot{x}^T (I_n \otimes A^T) (M \otimes I_3)^{-1} (I_n \otimes P) \dot{x} + \frac{1}{2} \dot{q}^T (M \otimes I_3)^T (M \otimes I_3)^{-1} (I_n \otimes P) \dot{x} \\ &\quad + \frac{1}{2} \dot{x}^T (M \otimes I_3)^{-1} (I_n \otimes P) (I_n \otimes A) \dot{x} + \frac{1}{2} \dot{x}^T (M \otimes I_3)^{-1} (I_n \otimes P) (M \otimes I_3) \dot{q} \\ &= - \frac{1}{2} \dot{x}^T (M \otimes I_3)^{-1} (I_n \otimes Q) \dot{x} + \dot{x}^T (I_n \otimes P) \dot{q} \end{aligned} \quad (6.63)$$

where (6.62) is used to obtain first equality, and the fact that $(M \otimes I_3)^{-1}$ commutes with both $I_n \otimes P$ and $I_n \otimes A^T$ is used to obtain the second equality.

Substituting (6.60) and (6.63) into (6.59) leads to

$$\begin{aligned}
 \dot{V} &= - \sum_{i=1}^n \tilde{\omega}_i^T S^T(\bar{q}_i) y_i - \frac{1}{2} \dot{x}^T (M \otimes I_3)^{-1} (I_n \otimes Q) \dot{x} + \dot{x}^T (I_n \otimes P) \dot{q} \\
 &= - \sum_{i=1}^n q_i^T (P \dot{x}_i) - \frac{1}{2} \dot{x}^T (M \otimes I_3)^{-1} (I_n \otimes Q) \dot{x} + \dot{x}^T (I_n \otimes P) \dot{q} \\
 &= - \frac{1}{2} \dot{x}^T (M \otimes I_3)^{-1} (I_n \otimes Q) \dot{x}
 \end{aligned} \tag{6.64}$$

It follows that $\dot{V} \leq 0$ because $(M \otimes I_3)^{-1} (I_n \otimes Q) = (I_n \otimes Q)^{-1} (M \otimes I_3)$ is a symmetric positive definite matrix.

Let $S = \{\tilde{\omega}_i, q_i, \dot{x}_i \mid \dot{V} = 0\}$, $i = 1, \dots, n$. Note that $\dot{V} \equiv 0$ implies $\dot{x} \equiv 0$, so is $\ddot{x} \equiv 0$. It follows $(M \otimes I_3) \dot{q} \equiv 0$ according to (6.62). Noting that $M \otimes I_3$ has full rank leads to $\dot{q} \equiv 0$. Thus, $\dot{q}_i \equiv 0$ using that fact $\bar{q}_i^2 + q_i^2 \equiv 0$. It follows that $\tilde{\omega}_i \equiv 0$ according to (6.46) and $A(\bar{q}_i)^T A(\bar{q}_i) = I_3$. From (6.11), (6.51), and the fact $y_i = P \dot{x}_i \equiv 0$, the following equations can be obtained

$$-\rho_i q_i - \sum_{j=1}^n \rho_{ij} q_{ij} \equiv 0, \quad i = 1, \dots, n \tag{6.65}$$

Following a similar procedure as in Lawton and Beard (2002), Abdessameud and Tayebi (2009), and using (6.3) and the definition of q_{ij} , the above equation can be written as follows:

$$(N \otimes I_3) q \equiv 0 \tag{6.66}$$

where $N = [n_{ij}] \in \mathbb{R}^{n \times n}$ defined as $n_{ii} = \rho_i + \sum_{j=1}^n \rho_{ij} q_{0,j}$ and $n_{ij} = -\rho_{ij} q_{0,j}$.

If the condition (6.53) is satisfied, the matrix N is strictly dominant due to $|q_{0,j}| \leq 1$, $j = 1, \dots, n$. Thus, N has full rank. It follows $q \equiv 0$ from (6.66). Thus, $q_i \equiv 0$. By LaSalle's invariance principle, it follows that

$$\lim_{t \rightarrow \infty} \tilde{\omega}_i(t) = \lim_{t \rightarrow \infty} q_i(t) = 0, \quad i = 1, \dots, n \tag{6.67}$$

Furthermore, as observed from (6.66), if $q_{0,i}(t) > 0$ for $i = 1, \dots, n$ is positive, then the matrix N is strictly dominant. Then, if there exist a time $T > 0$ such that $q_{0,i}(t) > 0$ for all $t \geq T$ and $i = 1, \dots, n$, then the same convergence results are obtained (Abdessameud and Tayebi 2009) without (6.53).

6.6 Simulation Studies

Simulation is carried out in this section to illustrate the performance and stability characteristics of the proposed decentralized coordinated attitude control laws. A scenario with four satellites is considered in the simulation. The inter-satellite undirected communication topology used is shown in Fig. 6.1. Although the communication graphs used in the simulation have a spanning tree (see Ren (2009)) for the definition of spanning tree), the proposed control laws are valid when there is no spanning tree. The corresponding weighted Laplacian matrix for the topology is

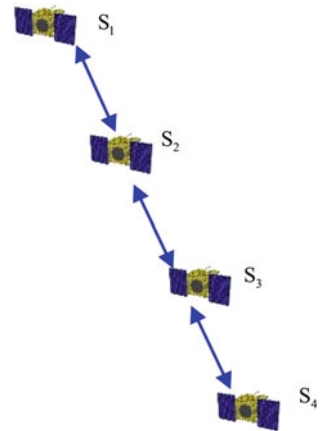
$$L = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -0.5 & 1 & -0.5 & 0 \\ 0 & -0.5 & 1 & -0.5 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

The actual inertia matrices of the satellites are assumed to be as follows (the unit of which is kg m^2)

$$\begin{aligned} J_1 &= \begin{bmatrix} 20 & 2 & 0.9 \\ 2 & 17 & 0.5 \\ 0.9 & 0.5 & 15 \end{bmatrix}, \quad J_2 = \begin{bmatrix} 22 & 1 & 0.9 \\ 1 & 19 & 0.5 \\ 0.9 & 0.5 & 15 \end{bmatrix}, \quad J_3 = \begin{bmatrix} 18 & 1 & 1.5 \\ 1 & 15 & 0.5 \\ 1.5 & 0.5 & 17 \end{bmatrix}, \\ J_4 &= \begin{bmatrix} 18 & 1 & 1 \\ 1 & 20 & 0.5 \\ 1 & 0.5 & 15 \end{bmatrix} \end{aligned} \quad (6.68)$$

The following two subsections present simulation results of adaptive sliding mode control in Sect. 6.4 and velocity-free coordinated attitude control in Sect. 6.5 for attitude synchronization and tracking, respectively.

Fig. 6.1 Undirected communication topology for satellite attitude synchronization



6.6.1 Decentralized Adaptive Sliding Mode Control

In order to validate the robustness against model uncertainties and external disturbances of the proposed control laws, it is assumed that the nominal inertia matrices of the satellites are

$$\bar{J}_1 = \bar{J}_2 = \bar{J}_3 = \bar{J}_4 = \text{diag}([20, 20, 20]^T) \text{kg m}^2 \quad (6.69)$$

and different sinusoidal-wave disturbances as in (6.70) are added to each satellite randomly. The external disturbances used in the simulation are far worse than those observed in practice.

$$\begin{aligned} d_1(t) &= [0.10 \sin(0.4t), 0.05 \cos(0.5t), 0.08 \cos(0.7t)]^T \text{Nm} \\ d_2(t) &= [0.06 \cos(0.4t), 0.10 \sin(0.5t), 0.05 \sin(0.7t)]^T \text{Nm} \\ d_3(t) &= [0.08 \sin(0.4t + \pi/4), 0.06 \cos(0.5t + \pi/4), 0.07 \cos(0.7t + \pi/4)]^T \text{Nm} \\ d_4(t) &= [0.06 \cos(0.4t + \pi/4), 0.08 \sin(0.5t + \pi/4), 0.10 \sin(0.7t + \pi/4)]^T \text{Nm} \end{aligned} \quad (6.70)$$

The initial angular velocity errors of all satellites are chosen to be zeros, and the initial attitude tracking errors are chosen as

$$\begin{aligned} \bar{q}_1(0) &= [0.8986 \quad 0.4 \quad -0.1 \quad 0.15]^T, \quad \bar{q}_2(0) = [0.8888 \quad -0.2 \quad 0.1 \quad 0.4]^T, \\ \bar{q}_3(0) &= [0.8062 \quad 0.1 \quad -0.5 \quad 0.3]^T, \quad \bar{q}_4(0) = [0.8426 \quad -0.4 \quad -0.2 \quad 0.3]^T. \end{aligned} \quad (6.71)$$

The initial desired attitudes are $\bar{\eta}_i^d(0) = [1 \quad 0 \quad 0 \quad 0]^T, i = 1, 2, 3, 4$. The time-varying desired angular velocities of the satellites are identical as follows:

$$\omega_i^d(t) = [0.1 \cos(t/40) \quad -0.1 \sin(t/50) \quad -0.1 \cos(t/60)]^T \quad i = 1, 2, 3, 4 \quad (6.72)$$

The controller parameters for the first control law (6.29) are chosen as $C = I_3$, $K_i = 0.01I_3$, $b_i = 1$, $\phi_i = 0.01$, $K_i = 0.01I_3$, and the parameters of the adaptation law (6.42) are chosen as $\kappa_{i,0} = 0.1$, $\kappa_{i,1} = \kappa_{i,2} = 0.2$ and $\hat{d}_{i,0}^0 = \hat{d}_{i,1}^0 = \hat{d}_{i,2}^0 = 0$, where $i = 1, \dots, n$.

The smoothing technique introduced in Sect. 6.4.3 is deployed for control law (6.29) to avoid the chattering of control input.

The attitude error, angular velocity error, and control torque of the 1st satellite are shown in Figs. 6.2, 6.3 and 6.4, respectively. For ease of interpretation, attitude errors are expressed by Euler angles converted from unit quaternion. It is observed that attitude tracking of the 1st satellite is achieved despite the presence of external

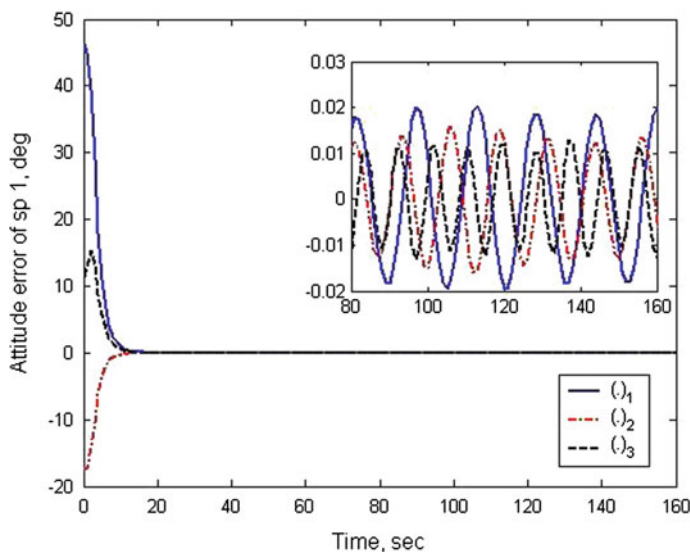


Fig. 6.2 Attitude tracking error of the 1st satellite under decentralized adaptive robust control with undirected communication topology

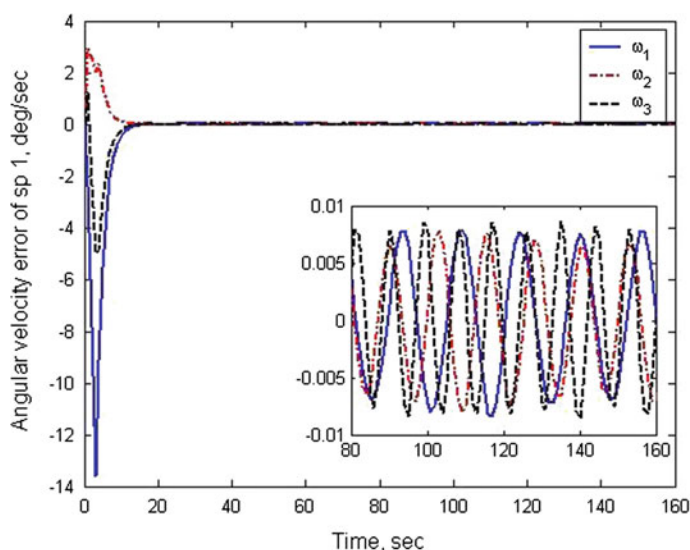


Fig. 6.3 Angular velocity error of the 1st satellite under decentralized adaptive robust control with undirected communication topology

disturbances and model uncertainties. The responses of the closed-loop systems of other satellites are similar to those of the 1st satellite and are not plotted here due to space constraint. The adaptive parameter $\hat{\alpha}_1$ is bounded as shown in Fig. 6.5, thus

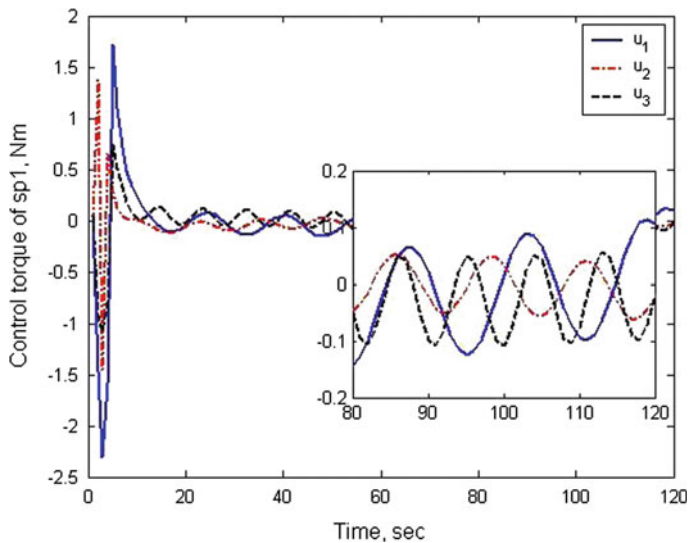
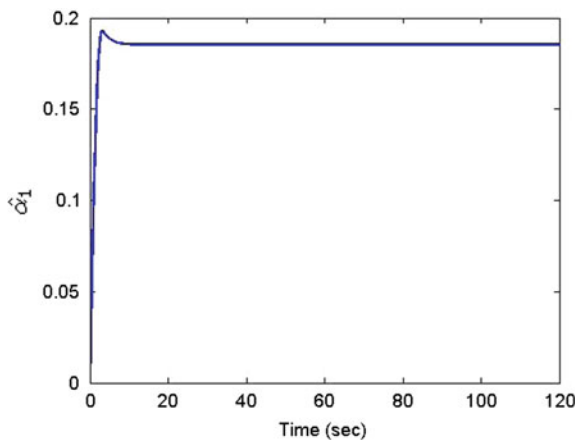


Fig. 6.4 Control torque of the 1st satellite under decentralized adaptive robust control with undirected communication topology

the proposed adaptation laws (6.42) are verified. The relative attitudes of the 2nd with respect to the 1st satellite and the 1st with respect to the 4th satellite are plotted in Figs. 6.6 and 6.7, respectively. Relative attitude errors between other pairs of satellites are similar to those of the above two.

For comparison, Figs. 6.8, 6.9, 6.10, and 6.11 show attitude error, relative attitude errors, and control torque when there is no coupling between neighbors [i.e., $a_{ij} = 0, i, j = 1, 2, 3, 4$ in (6.17)]. It is observed from Figs. 6.6 and 6.9 that relative attitude errors among satellites, especially steady-state relative attitude

Fig. 6.5 Adaptive parameter $\hat{\alpha}_1$ of the 1st satellite under decentralized adaptive robust control with undirected communication topology



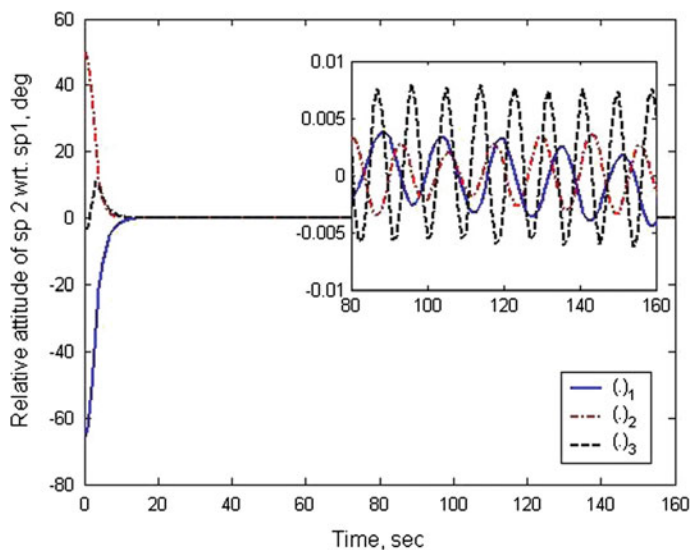


Fig. 6.6 Relative attitude errors of the 2nd satellite with respect to the 1st satellite under decentralized adaptive robust control with undirected communication topology

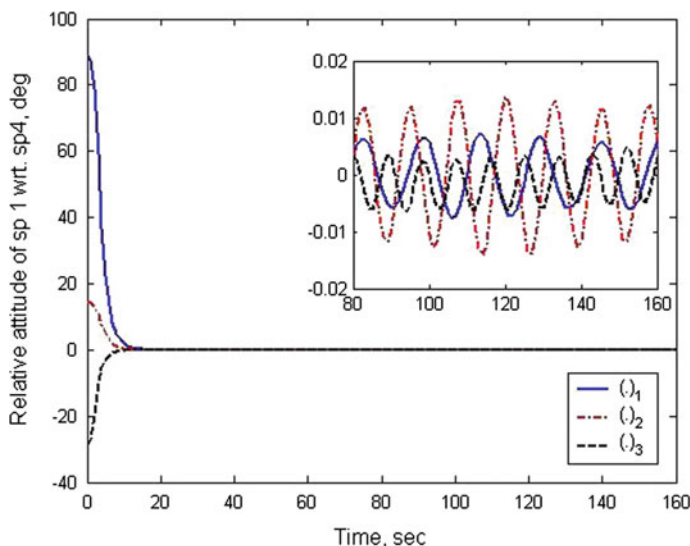


Fig. 6.7 Relative attitude errors of the 1st satellite with respect to the 4th satellite under decentralized adaptive robust control with undirected communication topology

errors, in the case with coupling between neighbors are reduced as compared to the case without coupling. Furthermore, even the relative attitude errors between satellites without direct inter-satellite communication are reduced, such as relative

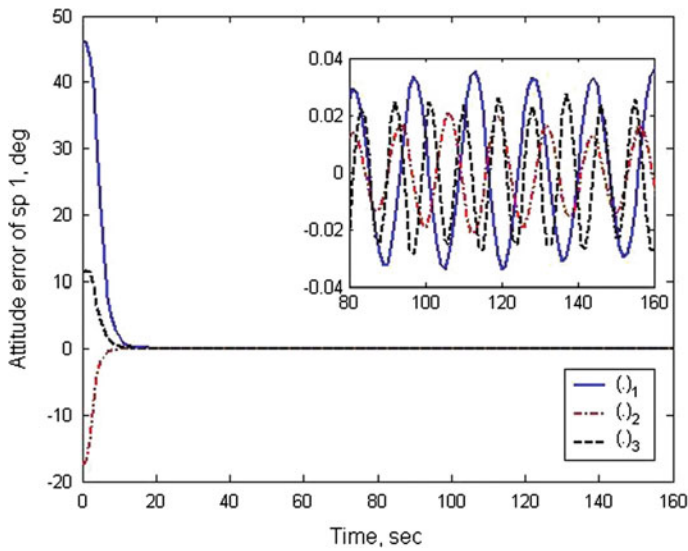


Fig. 6.8 Attitude tracking error of the 1st satellite without coupling between neighbors under decentralized adaptive robust control with undirected communication topology

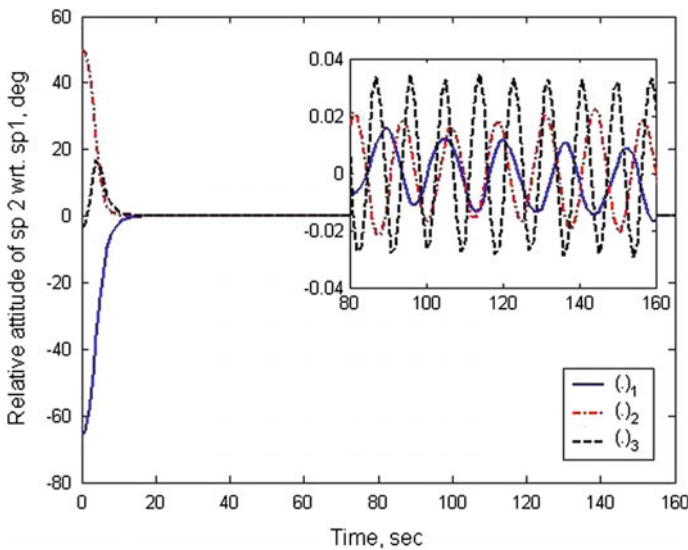


Fig. 6.9 Relative attitude error from the 1st to the 2nd satellite without coupling between neighbors under decentralized adaptive robust control with undirected communication topology

attitude errors between the 1st and 4th satellite as seen from Figs. 6.7 and 6.10. Although steady-state relative attitude errors are greatly reduced in the case with coupling, the control torques in the steady-state process for the two cases are almost

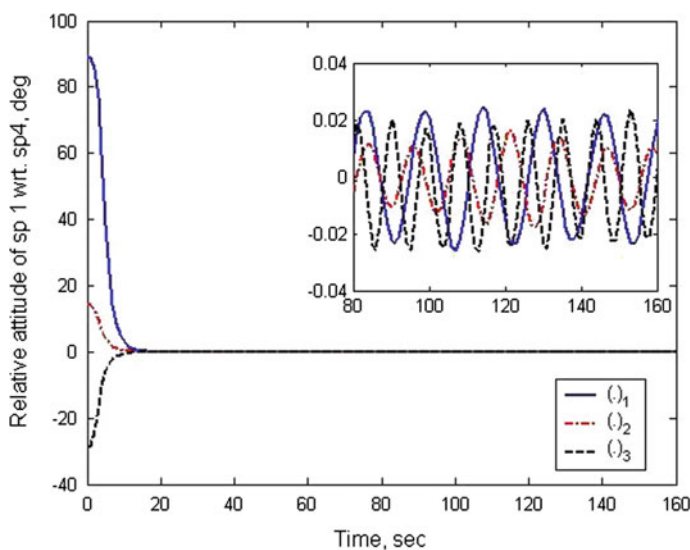


Fig. 6.10 Relative attitude error from the 4th to the 1st satellite without coupling between neighbors under decentralized adaptive robust control with undirected communication topology

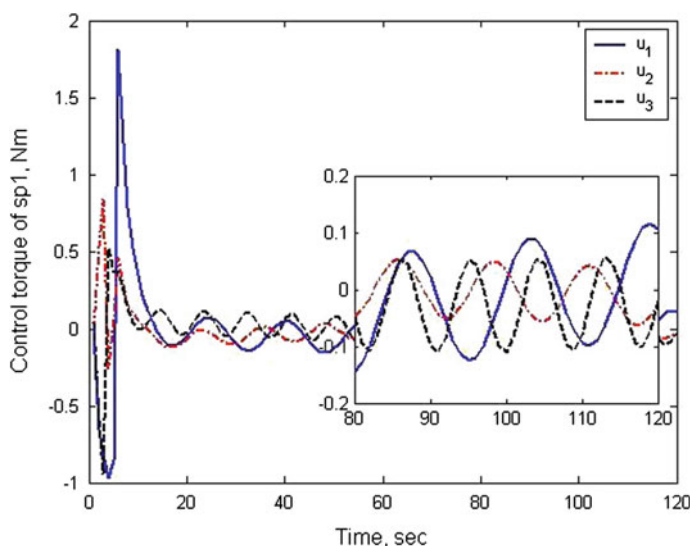


Fig. 6.11 Control torque of the 1st satellite without coupling between neighbors under decentralized adaptive robust control with undirected communication topology

identical, as can be seen from Figs. 6.4 and 6.11. Thus, simulation results validate the effectiveness of the proposed decentralized adaptive sliding mode control law for attitude synchronization and tracking.

6.6.2 Velocity-Free Coordinated Attitude Control

The initial angular velocities of the satellites are $\omega_i(0) = [-0.1 \ 0.1 \ 0.1]^T$ rad/s, $i = 1, 2, 3, 4$, and the initial attitudes are given as

$$\begin{aligned}\bar{\eta}_1(0) &= [0.8986, 0.4, -0.1, 0.15]^T, \\ \bar{\eta}_2(0) &= [0.8888, -0.2, 0.1, 0.4]^T, \\ \bar{\eta}_3(0) &= [0.8062, 0.1, -0.5, 0.3]^T, \\ \bar{\eta}_4(0) &= [0.8426, -0.4, -0.2, 0.3]^T.\end{aligned}\tag{6.73}$$

The initial desired quaternions are $\bar{\eta}_i^d(0) = [1 \ 0 \ 0 \ 0]^T$ and the desired angular velocities are $\omega_i^d(t) = [0.1 \cos(t/40), -0.1 \sin(t/50), -0.1 \cos(t/60)]^T$ rad/s, where $i = 1, 2, 3, 4$. Different square-wave disturbances are added to each satellite randomly as shown in Table 6.1. The external disturbances used in the simulation are worse than those observed in practice.

The controller parameters are chosen as, $A = -I_3$, $P = 8I_3$, $\rho_i = 2$, $k_i = 1$, $i = 1, 2, 3, 4$. Since $q_{0,i} > 0$, the condition (6.53) is not needed. Thus, in order to obtain better performance for attitude synchronization, the relative weight for attitude synchronization can be chosen larger as follows. $\rho_{12} = \rho_{43} = 4$, $k_{12} = k_{43} = 2$. $\rho_{21} = \rho_{23} = \rho_{32} = \rho_{34} = 2$, $k_{21} = k_{23} = k_{32} = k_{34} = 1$, other parameters of ρ_{ij} , k_{ij} are chosen to be zero according to the communication topology.

For comparison, two cases are considered in the simulation. In the first case, the coupling between neighbors corresponding to inter-satellite communication topology in Fig. 6.1 is used. In the second case, it is assumed there is no coupling between neighbors (i.e., $k_{ij} = \rho_{ij} = 0$ for all i and j).

Attitude tracking error, control torque of the 1st satellite, and relative attitude error of the 2nd with respect to the 1st satellite in the first case are shown in Figs. 6.12, 6.13 and 6.14, respectively. Attitude tracking errors of other satellites and relative attitude errors between other pairs of satellites are similar to those of the above. Due to space constraint, they are not plotted here. For ease of interpretation, attitude error is expressed by Euler angles converted from unit quaternion. It is observed from Figs. 6.12, 6.13 and 6.14 that attitude tracking of the 1st

Table 6.1 Square-wave disturbance of each satellite

	Period, s			Magnitude, Nm		
	Axis 1	Axis 2	Axis 3	Axis 1	Axis 2	Axis 3
Sp1	20	30	40	0.005	0.003	0.004
Sp2	40	15	20	0.003	0.005	0.007
Sp3	15	20	30	0.004	0.007	0.005
Sp4	30	25	15	0.003	0.004	0.007

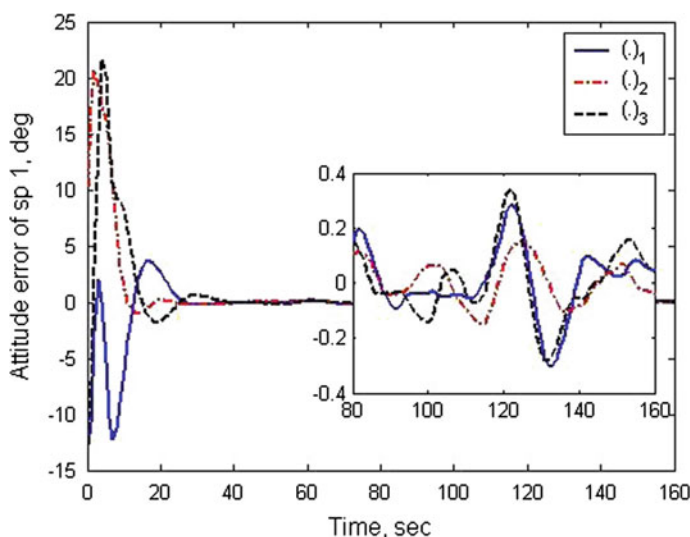


Fig. 6.12 Attitude tracking error of the 1st satellite under velocity-free coordinated attitude control with undirected communication topology

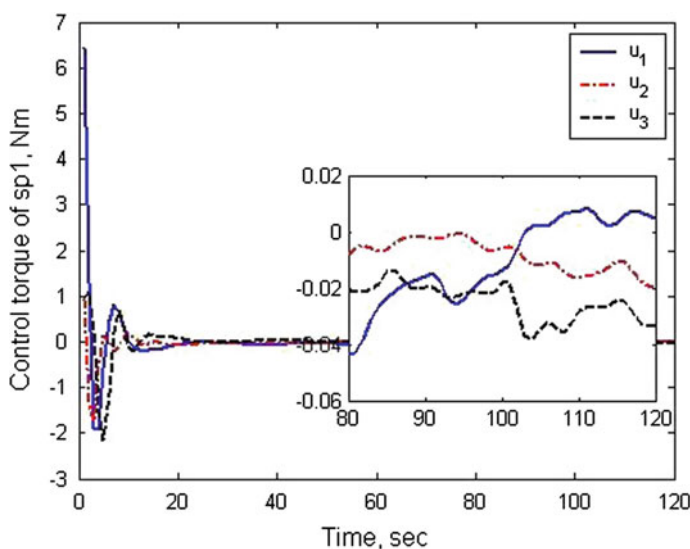


Fig. 6.13 Control torque of the 1st satellite under velocity-free coordinated attitude control with undirected communication topology

satellite is achieved, while the attitudes of the 1st and 2nd satellite are synchronized during the transition, despite the presence of external disturbances and the absence of angular velocity measurements.

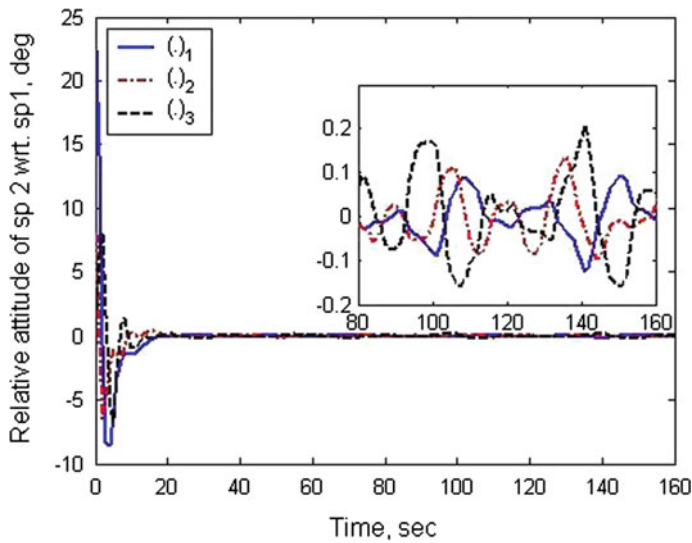


Fig. 6.14 Relative attitude error from the 1st to the 2nd satellite under velocity-free coordinated attitude control with undirected communication topology

Attitude tracking error, control torque of the 1st satellite and relative attitude error of the 2nd with respect to the 1st satellite in the second case are shown in Figs. 6.15, 6.16 and 6.17, respectively. As observed from Figs. 6.14 and 6.17,

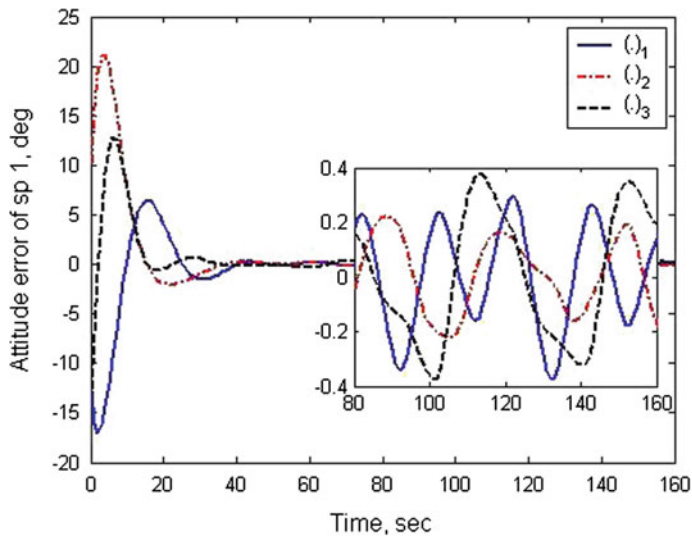


Fig. 6.15 Attitude tracking error of the 1st satellite without coupling between neighbors under velocity-free coordinated attitude control with undirected communication topology

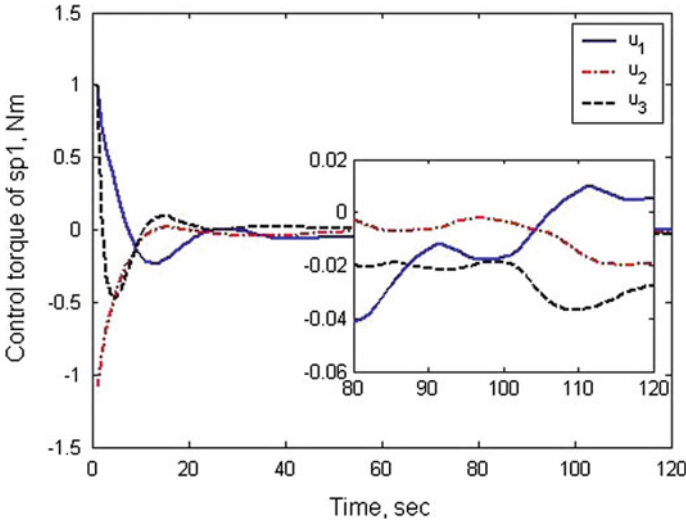


Fig. 6.16 Control torque of the 1st satellite without coupling between neighbors under velocity-free coordinated attitude control with undirected communication topology

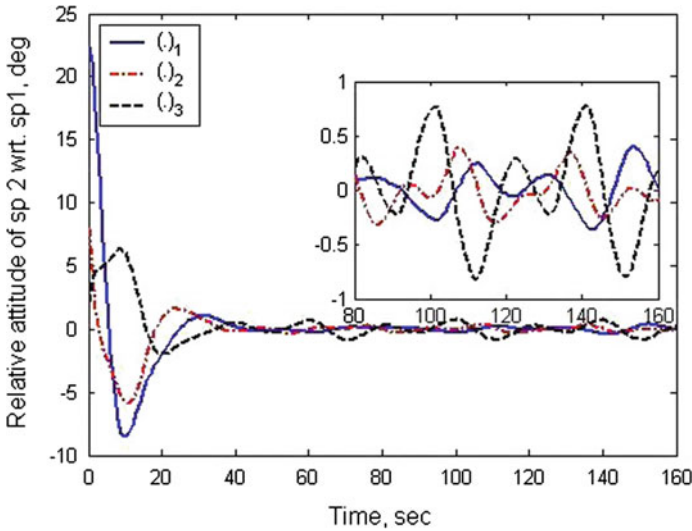


Fig. 6.17 Relative attitude error from the 1st to the 2nd satellite without coupling between neighbors under velocity-free coordinated attitude control with undirected communication topology

relative attitude errors among satellites are greatly reduced in the first case as compared to the second case, especially steady-state relative attitude errors. Steady-state relative attitude errors are reduced by 75 % approximately. Although

steady-state relative attitude errors in the first case are greatly reduced as compared to the second case, the magnitudes of control torques in the steady-state process for the two cases are almost identical, as can be seen from Figs. 6.13 and 6.16.

6.7 Conclusions

Attitude synchronization and tracking problems of satellite formation with undirected communication topology are studied with unit quaternion parameterization. Firstly, a decentralized adaptive sliding mode control law is designed by introducing the multi-satellite sliding mode vector, which includes attitude error, angular velocity error of each individual satellite, relative attitude errors, and relative angular velocity errors among satellites. Then, this chapter solves the attitude tracking synchronization problem without angular velocity measurements. The passivity approach is applied to generate the necessary damping in the absence of the actual absolute angular velocities measurements and relative angular velocities measurements between neighboring satellites. Numerical simulations are performed to validate the effectiveness of the proposed control laws. Simulation results demonstrate that each individual satellite converges to the desired attitude and angular velocity with fine control performance. Furthermore, relative attitude errors among satellites, especially steady-state relative attitudes errors, are reduced by including the relative attitude errors and relative angular velocity errors (relative damping in the second control law) in the proposed control laws.

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Chapter 7

Decentralized Control for Attitude Synchronization Under Directed Communication Topology

This chapter developed two decentralized coordinated attitude control laws for attitude synchronization and tracking problem with directed inter-satellite communication topology. Firstly, an adaptive sliding mode control law is developed against inertia matrix uncertainties and external disturbances. Then, an adaptive backstepping control law with considering inter-satellite communication delay is proposed.

7.1 Introduction

In previous chapter, two decentralized coordinated attitude control laws are proposed for the undirected inter-satellite communication topology. However, in practice, the inter-satellite communication topology may be restricted to be directed, such as in unidirectional satellite laser communication system. Thus, two decentralized adaptive cooperative attitude control laws are developed for attitude synchronization with directed inter-satellite communication topology. Firstly, a robust adaptive control law is developed against inertia matrix uncertainties and external disturbances. Time delays exist in the inter-satellite communication channels, which can lead to substantial performance deterioration and even to instability of the attitude control system. Hence, an adaptive backstepping control law considering communication delay is subsequently proposed. The reported simulation results confirm the effectiveness of the proposed methods with directed inter-satellite communication topology.

7.2 Decentralized Adaptive Robust Control for Attitude Synchronization

In this section, a decentralized adaptive sliding mode control law is proposed for attitude synchronization and tracking using a general directed communication topology (Wu et al. 2011). Firstly, the multi-satellite sliding mode vector is developed. Subsequently, a decentralized adaptive sliding mode control law is derived based on the multi-satellite sliding mode vector.

Assumptions 6.1 and 6.3–6.5 are also made about the dynamics of the attitude synchronization and tracking systems in this section

7.2.1 Multi-satellite Sliding Manifold

In Sect. 6.4.1, a multi-satellite sliding manifold is designed for the case of undirected communication topology. In this subsection, the multi-satellite sliding mode vector is modified for the case of directed communication topology and is defined as

$$S = [s_1, \dots, s_n]^T \quad (7.1)$$

where $s_i \in \mathbb{R}^{3 \times 1}$ is given by

$$s_i = b_i \bar{J}_i (\tilde{\omega}_i + Cq_i) + \sum_{j=1, j \neq i}^n a_{ij} [(\bar{J}_i \tilde{\omega}_i - \bar{J}_j \tilde{\omega}_j) + (\bar{J}_i Cq_i - \bar{J}_j Cq_j)], \quad i = 1, \dots, n. \quad (7.2)$$

with C being a positive definite constant matrix, scalar $b_i > 0$ is the control weight parameter for attitude tracking of the i th satellite (station-keeping behavior), scalar $a_{ij} \geq 0$ defined in (A.1) is the control weight parameter for inter-satellite attitude synchronization between the i th and j th satellites (formation-keeping behavior), and $\bar{J}_i$ is the nominal inertia matrix of the i th satellite.

Remark 7.1 In the case of undirected communication topology, it is valid to assume $a_{i,j} = a_{j,i}$ in (7.2). This assumption greatly simplifies the stability proof of the cooperative attitude control system, because some terms involving $a_{i,j}$ in the time derivative of the Lyapunov function are eliminated by mutual cancelation. However, it is not valid to assume $a_{i,j} = a_{j,i}$ in the case of directed communication topology. Thus, cooperative control for attitude synchronization under directed communication topology is more challenging as compared to the case of undirected communication topology.

Using the Kronecker product, the multi-satellite sliding vector (7.1) can be rewritten as

$$S = [(L+B) \otimes I_3] \bar{\mathbf{J}} (\tilde{\Omega} + \tilde{C}Q) \quad (7.3)$$

where $\tilde{\Omega} = [\tilde{\omega}_1, \dots, \tilde{\omega}_n]^T$, $Q = [q_1, \dots, q_n]^T$, $B = \text{diag} [b_1, \dots, b_n]$, I_3 denotes the 3×3 identity matrix, $\tilde{C} = \text{diag}[C, \dots, C] \in \mathbb{R}^{3n \times 3n}$, L is the weighted Laplacian matrix corresponding to the inter-satellite directed communication topology as described in (A.3), and $\bar{\mathbf{J}} = \text{diag}[\bar{J}_1, \dots, \bar{J}_n]$.

The multi-satellite sliding mode surface is then defined as $S = 0$. In view of Proposition A.3, Lemma 5.1(iii) and the definition of B , it follows that $(L+B) \otimes I_3$ has full rank. Furthermore, $\bar{\mathbf{J}}$ has full rank. Consequently, $[(L+B) \otimes I_3] \bar{\mathbf{J}}$ has full rank. Thus, it follows that $\tilde{\Omega} + \tilde{C}Q = 0$ on the sliding mode surface $S = 0$ from (7.3). $\tilde{\Omega} + \tilde{C}Q = 0$ implies $\tilde{\omega}_i + Cq_i = 0$, $i = 1, \dots, n$. It is obvious that $\tilde{\omega}_i + Cq_i = 0$ implies that

$$\lim_{t \rightarrow \infty} \|\tilde{\omega}_i\| = \lim_{t \rightarrow \infty} \|q_i\| = 0, \quad i = 1, \dots, n \quad (7.4)$$

It can be concluded that on the multi-satellite sliding mode surface $S = 0$, the attitude synchronization and tracking problem in Definition 6.1 is asymptotically solved.

7.2.2 Decentralized Adaptive Sliding Mode Control Design

This subsection presents the adaptive sliding mode control law which ensures that the satellite attitude error dynamics (6.11)–(6.13) will converge to the sliding manifold $S = 0$.

To develop the control law, the following equations are derived from (6.11) and (6.12)

$$\bar{J}_i(\dot{\tilde{\omega}}_i + C\dot{q}_i) = h_i + \rho_i + u_i, \quad i = 1, \dots, n \quad (7.5)$$

with

$$h_i(t) \triangleq -\omega_i^\times \bar{J}_i \omega_i + \bar{J}_i(\tilde{\omega}_i^\times R(\bar{q}_i) \omega_i^d - R(\bar{q}_i) \dot{\omega}_i^d) + \frac{1}{2} \bar{J}_i C(q_i^\times + q_{i,0} I) \tilde{\omega}_i \quad (7.6)$$

$$\rho_i(t) \triangleq d_i - \Delta J_i \dot{\tilde{\omega}}_i - \omega_i^\times \Delta J_i \omega_i + \Delta J_i(\tilde{\omega}_i^\times R(\bar{q}_i) \omega_i^d - R(\bar{q}_i) \dot{\omega}_i^d) \quad (7.7)$$

$\rho_i(t)$ represents the inertia matrix uncertainties and external disturbances. Under Assumptions 6.1 and 6.3–6.5, and the definition of L and B , it can be shown that

$$\|(L+B) \otimes I_3\|_1 \|\rho_i(t)\|_1 \leq c_{i,0} + c_{i,1} \sum_{j \in N_i} \|\tilde{\omega}_j\|_1 + c_{i,2} \sum_{j \in N_i} \|\tilde{\omega}_j\|_1^2 \quad (7.8)$$

where $c_{i,0}$, $c_{i,1}$, and $c_{i,2}$, are nonnegative constant numbers.

However, as mentioned in Sect. 6.4.2, in the conventional sliding mode control laws for attitude control in (Lo and Chen 1995; Crassidis et al. 2000; Boškovic et al. 2001), an important assumption is that inertia uncertainties and external disturbances are bounded and that their bounds are known to the designer. The implementation of the control law, based on a conservative bound, may result in impractically large control authority and control chattering. Therefore, in order to avoid the requirement of prior knowledge of the upper bound of $\|(L+B) \otimes I_3\|_1 \|\rho_i(t)\|_1$, an adaptive mechanism is introduced to estimate its upper bound parameters.

Let $\hat{c}_{i,0}$, $\hat{c}_{i,1}$, and $\hat{c}_{i,2}$ denote the estimates of $c_{i,0}$, $c_{i,1}$, and $c_{i,2}$, respectively. Now, consider the simple adaptation laws for the upper bound of the norm $\|(L+B) \otimes I_3\|_1 \|\rho_i(t)\|_1$ such that

$$\begin{aligned} \dot{\hat{c}}_{i,0} &\triangleq \kappa_{i,0} \|s_i\|_1 \\ \dot{\hat{c}}_{i,1} &\triangleq \kappa_{i,1} \|s_i\|_1 \sum_{j \in N_i} \|\tilde{\omega}_j\|_1 \\ \dot{\hat{c}}_{i,2} &\triangleq \kappa_{i,2} \|s_i\|_1 \sum_{j \in N_i} \|\tilde{\omega}_j\|_1^2 \end{aligned} \quad (7.9)$$

where $\tilde{c}_{i,0} = \hat{c}_{i,0} - c_{i,0}$, $\tilde{c}_{i,1} = \hat{c}_{i,1} - c_{i,1}$ and $\tilde{c}_{i,2} = \hat{c}_{i,2} - c_{i,2}$ are parameter adaptation errors, $\kappa_{i,0}$, $\kappa_{i,1}$ and $\kappa_{i,2}$ are positive adaptive gains, and s_i is the component of multi-satellite sliding mode vector S , which is defined in (7.2).

The control input $u_i \in \mathbb{R}^{3 \times 1}$ for the i th satellite is proposed as

$$\begin{aligned} u_i &= -h_i + \left(\sum_{j=1, j \neq i}^n a_{ij} + b_i \right)^{-1} \left[\sum_{j=1, j \neq i}^n a_{ij} (u_j + h_j) - K_i s_i - \hat{\beta}_i \operatorname{sgn}(s_i) \right], \quad i \\ &= 1, \dots, n \end{aligned} \quad (7.10)$$

where h_i is defined in (7.6), $K_i \in \mathbb{R}^{3 \times 3}$ is a positive definite gain matrix, a_{ij} and b_i are defined in (7.2), $\operatorname{sgn}(s_i) \triangleq [\operatorname{sgn}(s_{i,1}) \quad \operatorname{sgn}(s_{i,2}) \quad \operatorname{sgn}(s_{i,3})]^T$, $s_{i,j}$, $j = 1, 2, 3$ is the j th component of s_i , $\hat{\beta}_i$ is the adaptive upper bound of the norm $\|(L+B) \otimes I_3\|_1 \|\rho_i(t)\|_1$ and is defined by

$$\hat{\beta}_i \triangleq \hat{c}_{i,0} + \hat{c}_{i,1} \sum_{j \in N_i} \|\tilde{\omega}_j\|_1 + \hat{c}_{i,2} \sum_{j \in N_i} \|\tilde{\omega}_j\|_1^2, \quad i = 1, \dots, n \quad (7.11)$$

Note that in the proposed control law (7.10), the information flow between satellites includes quaternion error and angular velocity error, as well as control input.

The following theorem establishes the condition for the existence of the multi-satellite sliding mode surface $S = 0$ for satellite formation attitude tracking systems described by (6.11)–(6.13).

Theorem 7.1 *Consider the satellite formation attitude tracking dynamics described by (6.11)–(6.13) with the decentralized adaptive sliding mode control law (7.10) and the adaptation law (7.9). If Assumptions 6.1 and 6.3–6.5 are valid, then the multi-satellite sliding mode surface $S = 0$ defined in (7.1) will be reached asymptotically.*

Proof The candidate Lyapunov function is chosen as

$$V = V_1 + V_2 \quad (7.12)$$

with

$$V_1 = \frac{1}{2} S^T S$$

$$V_2 = \frac{1}{2} \sum_{i=1}^n \left(\kappa_{i,0}^{-1} \tilde{c}_{i,0}^2 + \kappa_{i,1}^{-1} \tilde{c}_{i,1}^2 + \kappa_{i,2}^{-1} \tilde{c}_{i,2}^2 \right)$$

Using the Kronecker product and the definition of adjacent matrix A and in-degree matrix D of inter-satellite communication graph, (7.10) can be rewritten as

$$U = -H + \left[(D+B)^{-1} \otimes I_3 \right] \left[(A \otimes I_3)(U+H) - \mathbf{K}S - \hat{\beta} \operatorname{sgn}(S) \right] \quad (7.13)$$

where $U = [u_1, \dots, u_n]^T$, $H = [h_1, \dots, h_n]^T$, $\mathbf{K} = \operatorname{diag}[K_1, \dots, K_n]$ and $\hat{\beta} = \operatorname{diag}[\hat{\beta}_1 I_3, \dots, \hat{\beta}_n I_3]$.

The above equation can be rewritten as

$$U = -H - \left\{ I_{3n} - \left[(D+B)^{-1} \otimes I_3 \right] (A \otimes I_3) \right\}^{-1} \left[(D+B)^{-1} \otimes I_3 \right] (\mathbf{K}S + \hat{\beta} \operatorname{sgn}(S))$$

$$= -H - [(L+B) \otimes I_3]^{-1} (\mathbf{K}S + \hat{\beta} \operatorname{sgn}(S)) \quad (7.14)$$

where the following equality is used

$$I_{3n} - \left[(D+B)^{-1} \otimes I_3 \right] (A \otimes I_3) = \left[(D+B)^{-1} \otimes I_3 \right] \{ [(D+B) \otimes I_3] - A \otimes I_3 \}$$

$$= \left[(D+B)^{-1} \otimes I_3 \right] [(L+B) \otimes I_3] \quad (7.15)$$

Taking the first derivative of V_1 and using (7.3) and (7.5) leads to

$$\begin{aligned}
\dot{V}_1 &= S^T \dot{S} \\
&= S^T [(L+B) \otimes I_3] (\bar{\mathbf{J}} \dot{\Omega} + \bar{\mathbf{J}} \tilde{C} \dot{Q}) \\
&= S^T [(L+B) \otimes I_3] (H + \rho + U)
\end{aligned} \tag{7.16}$$

where $\rho = [\rho_1, \dots, \rho_n]^T$.

Substituting (7.14) into the above equation yields

$$\begin{aligned}
\dot{V}_1 &= S^T [(L+B) \otimes I_3] \rho - S^T \hat{\beta} \operatorname{sgn}(S) - S^T \mathbf{K} S \\
&\leq \sum_{i=1}^n \left(-s_i^T K_i s_i - \hat{\beta}_i \|s_i\|_1 + \|(L+B) \otimes I_3\|_1 \cdot \|\rho_i\|_1 \cdot \|s_i\|_1 \right) \\
&\leq \sum_{i=1}^n \left[-s_i^T K_i s_i - \left(\hat{c}_{i,0} + \hat{c}_{i,1} \sum_{j \in N_i} \|\tilde{\omega}_j\|_1 + \hat{c}_{i,2} \|\tilde{\omega}_i\|_1^2 \right) \|s_i\|_1 \right] \\
&\quad + \sum_{i=1}^n \left[\|s_i\|_1 \cdot \left(c_{i,0} + c_{i,1} \sum_{j \in N_i} \|\tilde{\omega}_j\|_1 + c_{i,2} \sum_{j \in N_i} \|\tilde{\omega}_j\|_1^2 \right) \right] \\
&= \sum_{i=1}^n \left[-s_i^T K_i s_i - \left(\tilde{c}_{i,0} + \tilde{c}_{i,1} \sum_{j \in N_i} \|\tilde{\omega}_j\|_1 + \tilde{c}_{i,2} \sum_{j \in N_i} \|\tilde{\omega}_j\|_1^2 \right) \|s_i\|_1 \right]
\end{aligned} \tag{7.17}$$

Taking the first derivative of V_2 and making use of the parameter adaptation law (7.9) yields

$$\begin{aligned}
\dot{V}_2 &= \sum_{i=1}^n \left(\kappa_{i,0}^{-1} \tilde{c}_{i,0} \dot{\tilde{c}}_{i,0} + \kappa_{i,1}^{-1} \tilde{c}_{i,1} \dot{\tilde{c}}_{i,1} + \kappa_{i,2}^{-1} \tilde{c}_{i,2} \dot{\tilde{c}}_{i,2} \right) \\
&= - \sum_{i=1}^n \left[\left(\tilde{c}_0 + \tilde{c}_1 \sum_{j \in N_i} \|\tilde{\omega}_j\|_1 + \tilde{c}_2 \sum_{j \in N_i} \|\tilde{\omega}_j\|_1^2 \right) \|s_i\|_1 \right]
\end{aligned} \tag{7.18}$$

Adding the above equation to (7.17) leads to

$$\dot{V} \leq - \sum_{i=1}^n s_i^T K_i s_i \leq 0 \tag{7.19}$$

Therefore, it follows that $s_i \in \mathcal{L}^\infty$, and $\tilde{c}_{i,0}, \tilde{c}_{i,1}, \tilde{c}_{i,2} \in \mathcal{L}^\infty$. Consequently, from (7.10) and Assumption 6.1, it is obtained that $u_i \in \mathcal{L}^\infty$. It follows that $\dot{\tilde{\omega}}_i, \dot{q}_i$ hence $\dot{s}_i$ are all bounded from (6.11) and (6.12). Integrating $\dot{V}$ gives the results that $s_i \in \mathcal{L}^2$. Hence, using the corollary of Barbalat's Lemma in (Sastry and Bodson 1989), it follows that $\lim_{t \rightarrow \infty} s_i(t) = 0, i = 1, \dots, n$. Thus, it is obtained that $\lim_{t \rightarrow \infty} S(t) = 0$. $\square$

Remark 7.2 There is no assumption on inter-satellite communication topology in the proposed control law. Thus, the proposed control law is applicable to any communication topology and is not restricted to be ring topology or undirected communication topology. The stability of the proposed control law is guaranteed even when there is no communication link, and then each satellite will be controlled individually.

Remark 7.3 Unlike (VanDyke and Hall 2006) and (Jin and Sun 2009), there is no additional restriction on parameters b_i and a_{ij} other than $b_i > 0$, $a_{ij} \geq 0$ in (7.10). So the proposed attitude synchronization and tracking scheme will allow the designer to prioritize between station-keeping behavior and formation-keeping behavior. For example, if one want to prioritize formation-keeping behavior, a_{ij} should be chosen to be large relative to b_i .

Remark 7.4 In the proposed control law, the desired attitude of each satellite with respect to inertial frame I is not restricted to be the same. Thus, the desired relative attitude between satellites can be maintained.

Remark 7.5 The proposed control law (7.10) is discontinuous across the surface $S(t)$, thus leading to control chattering. The smoothing control technique introduced in Sect. 6.4.3 can also be deployed to avoid the chattering of control input.

7.3 Decentralized Adaptive Backstepping Control for Attitude Synchronization with Communication Delay

Time delays exist in inter-satellite communication channels, which may make the whole control system lose stability. Hence, this section proposes an adaptive backstepping control approach for satellite attitude synchronization and tracking under directed communication topology with time delay.

Backstepping is a popular nonlinear control design technique (Krstić et al. 1995). The term backstepping refers to the recursive nature of the control design procedure in which a control law and a control Lyapunov function are recursively constructed to guarantee stability. The advantage of integrator backstepping compared with other control methods lies in its design flexibility, due to its recursive use of Lyapunov functions. The control torque is designed for each integrator level in the system and thus enables the possibility to compensate for destabilizing nonlinearities, while stabilizing nonlinearities can be exploited. Backstepping has been considered for the satellite attitude control (Krstić and Tsiotras 1999; Kristiansen et al. 2009; Ali et al. 2010). The cascaded structure of satellite kinematics and dynamics makes the integrator backstepping a preferred approach for the satellite attitude control problem. This section proposes a decentralized adaptive backstepping control law for attitude synchronization problem in the presence of inertia matrix uncertainties, external disturbances, and communication delays.

The following assumptions are made about the dynamics of the attitude synchronization and tracking systems

Assumption 7.1 The upper bound of norm of the external disturbances $d_i(t)$ is assumed to be known as

$$\|d_i(t)\| \leq D_i \quad (7.20)$$

Assumption 6.6 is also assumed to be valid in this section.

Observe that the attitude tracking error system in (6.11)–(6.13) is a nonlinear cascade interconnection, that is, the kinematics subsystem (6.12) and (6.13) is stabilized only indirectly through the angular velocity error vector $\tilde{\omega}_i$. Stabilizing control laws for cascade systems can be efficiently designed using the method of integrator backstepping (Krstić et al. 1995). By this method, $\tilde{\omega}_i$ in (6.11)–(6.13) is considered as a virtual control input of the i th satellite and a control law α_i is designed to stabilize the kinematics subsystem. Subsequently, the actual control u_i is designed to stabilize the dynamics subsystem (6.11) without destabilizing the kinematics subsystem (6.12) and (6.13).

Step (1) Synchronization of the kinematics subsystems: The first backstepping $\tilde{\omega}_i$ variable is chosen as $z_{1i} = [1 - q_{0,i}, q_i^T]^T$. Consider $\tilde{\omega}_i$ in the kinematics subsystem (6.12) and (6.13) as control input. Let z_{2i} be an error variable representing the difference between the actual and virtual control of the i th spacecraft for the kinematic subsystem (6.12) and (6.13), i.e. $z_{2i} = \tilde{\omega}_i - \alpha_i$. The virtual control law for the i th satellite is chosen as

$$\alpha_i = -\rho_i q_i - \sum_{j=1, j \neq i}^n \left(\rho_{ij}^a q_i - \rho_{ij}^b q_j(t - T_{ij}) \right), \quad i, j = 1, \dots, n \quad (7.21)$$

where T_{ij} denotes the nonnegative time-varying communication delay from the j th to the i th satellite, T_{ij} satisfies the following assumption.

Assumption 7.2 $T_{ij} = T_{ji}$, $\dot{T}_{ij} < 1$, $i, j = 1, \dots, n$, $i \neq j$

Remark 7.6 The virtual control input (7.21) for i th satellite makes use of the error quaternion q_j of its neighboring satellites. This information flow can be achieved by inter-satellite communication.

Remark 7.7 The term $-\rho_i q_i$ in (7.21) is for attitude tracking of each individual satellite, and the term $\sum_{j=1, j \neq i}^n (\rho_{ij}^a q_i - \rho_{ij}^b q_j(t - T_{ij}))$ is for attitude synchronization among satellites.

The control parameters in (7.21) satisfy the following constraints

$$\rho_i > 0, \rho_{ij}^a \geq 0, \rho_{ij}^b \geq 0 \quad (7.22)$$

$$\rho_{ij}^b \leq \sqrt{4\gamma_{ij}(1 - \dot{T}_{ij}) \left(c_i \rho_{ij}^a - \gamma_{ij} \right) / c_i^2} \quad (7.23)$$

where c_i is a positive constant, γ_{ij} is an arbitrary positive constant which is not used in control law and only used in stability proof, thus γ_{ij} can be chosen to guaranteed (7.23) is satisfied.

The candidate of Lyapunov function is chosen as

$$V_1 = \sum_{i=1}^n c_i z_{1i}^T z_{1i} + \sum_{i=1}^n \sum_{j=1}^n \gamma_{ij} \int_{t-T_{ij}}^t (q_i(\tau))^T q_i(\tau) d\tau \quad (7.24)$$

where the constant $c_i > 0$. The derivative of V_1 is given by

$$\begin{aligned} \dot{V}_1 &= \sum_{i=1}^n c_i q_i^T \tilde{\omega}_i + \sum_{i=1}^n \sum_{j=1}^n \gamma_{ij} \left[q_i^T q_i - (1 - \dot{T}_{ij}) (q_i(t - T_{ij}))^T (q_i(t - T_{ij})) \right] \\ &= \sum_{i=1}^n c_i q_i^T \left(z_{2i} - \rho_i q_i - \sum_{j=1}^n \left(\rho_{ij}^a q_i - \rho_{ij}^b q_j(t - T_{ij}) \right) \right) \\ &\quad + \sum_{i=1}^n \sum_{j=1}^n \gamma_{ij} \left[q_i^T q_i - (1 - \dot{T}_{ij}) (q_i(t - T_{ij}))^T (q_i(t - T_{ij})) \right] \\ &= - \sum_{i=1}^n c_i \rho_i q_i^T q_i - \sum_{i=1}^n \sum_{j=1}^n \left(c_i \rho_{ij}^a - \gamma_{ij} \right) q_i^T q_i + \sum_{i=1}^n \sum_{j=1}^n c_i \rho_{ij}^b q_i^T q_j(t - T_{ij}) \\ &\quad - \sum_{i=1}^n \sum_{j=1}^n \gamma_{ij} (1 - \dot{T}_{ij}) (q_i(t - T_{ij}))^T (q_i(t - T_{ij})) + \sum_{i=1}^n c_i q_i^T z_{2i} \end{aligned} \quad (7.25)$$

If $T_{ij} = T_{ji}$ and $\gamma_{ij} = \gamma_{ji}$, then the above equation can be written as

$$\begin{aligned} \dot{V}_1 &= - \sum_{i=1}^n c_i \rho_i q_i^T q_i - \sum_{i=1}^n \sum_{j=1}^n \left(c_i \rho_{ij}^a - \gamma_{ij} \right) q_i^T q_i + \sum_{i=1}^n \sum_{j=1}^n c_i \rho_{ij}^b q_i^T q_j(t - T_{ij}) \\ &\quad - \sum_{i=1}^n \sum_{j=1}^n \gamma_{ij} (1 - \dot{T}_{ij}) (q_j(t - T_{ij}))^T (q_j(t - T_{ij})) + \sum_{i=1}^n c_i q_i^T z_{2i} \end{aligned} \quad (7.26)$$

The above equation can be rewritten as

$$\begin{aligned} \dot{V}_1 = & - \sum_{i=1}^n c_i \rho_i q_i^T q_i - \sum_{i=1}^n \sum_{j=1}^n \varepsilon_{ij}^T \varepsilon_{ij} \sum_{i=1}^n c_i q_i^T z_{2i} \\ & - \sum_{i=1}^n \sum_{j=1}^n \left[\gamma_{ij} (1 - \dot{T}_{ij}) - \frac{(c_i \rho_{ij}^b)^2}{4(c_i \rho_{ij}^a - \gamma_{ij})} \right] (q_j(t - T_{ij}))^T (q_j(t - T_{ij})) \end{aligned} \quad (7.27)$$

where

$$\varepsilon_{ij} = \sqrt{c_i \rho_{ij}^a - \gamma_{ij}} q_i - \frac{c_i \rho_{ij}^b}{2\sqrt{c_i \rho_{ij}^a - \gamma_{ij}}} (q_j(t - T_{ij})) \quad (7.28)$$

Note that (7.23) yields

$$\gamma_{ij} (1 - \dot{T}_{ij}) - \frac{(c_i \rho_{ij}^b)^2}{4(c_i \rho_{ij}^a - \gamma_{ij})} \geq 0 \quad (7.29)$$

Thus, combining (7.27) with (7.29) leads to

$$\dot{V}_1 = - \sum_{i=1}^n \rho_i q_i^T q_i - \sum_{i=1}^n \sum_{j=1}^n \varepsilon_{ij}^T \varepsilon_{ij} + \sum_{i=1}^n c_i q_i^T z_{2i} \quad (7.30)$$

Clearly, if $z_{2i} = 0$, then $\dot{V}_1 = - \sum_{i=1}^n \rho_i q_i^T q_i - \sum_{i=1}^n \sum_{j=1}^n \varepsilon_{ij}^T \varepsilon_{ij}$ and z_{1i} is guaranteed to converge to zero asymptotically.

Step (2) Synchronization of the full rigid-body models:

From (7.21), it follows that

$$z_{2i} = \tilde{\omega}_i - \alpha_i = \tilde{\omega}_i + \rho_i^p q_i + \sum_{j=1}^n \left(\rho_{ij}^a q_i - \rho_{ij}^b q_j(t - T_{ij}) \right) \quad (7.31)$$

By using (6.12) and (6.13) and the above equation, it is obtained that

$$\begin{aligned} J_i \dot{z}_{2i} = & -\omega_i^\times J_i \omega_i + J_i (\tilde{\omega}_i^\times R(\bar{q}_i) \omega_i^d - R(\bar{q}_i) \dot{\omega}_i^d) + u_i \\ & + \frac{1}{2} \rho_i J_i S(\bar{q}_i) \tilde{\omega}_i + \frac{1}{2} J_i \sum_{j=1}^n \left[\rho_{ij}^a S(\bar{q}_i) \tilde{\omega}_i - \rho_{ij}^b S(\bar{q}_j(t - T_{ij})) \tilde{\omega}_j(t - T_{ij}) \right] \end{aligned} \quad (7.32)$$

Following the approach in (Ahmed et al. 1998), the following relationship holds

$$J_i x = L(x) \beta_i \quad (7.33)$$

where $x = [x_1 \ x_2 \ x_3]^T$ and

$$L(x) = \begin{bmatrix} x_1 & 0 & 0 & 0 & x_3 & x_2 \\ 0 & x_2 & 0 & x_3 & 0 & x_1 \\ 0 & 0 & x_3 & x_2 & x_1 & 0 \end{bmatrix} \quad (7.34)$$

and $\beta_i \in \mathbb{R}^6$ represents the unknown constant parameters of inertia matrix, and is defined as

$$\beta_i = [J_{i,11} \ J_{i,22} \ J_{i,33} \ J_{i,23} \ J_{i,13} \ J_{i,12}]^T \quad (7.35)$$

By using (7.33), (7.32) can be rewritten as

$$J_i \dot{z}_{2i} = F_i(\bar{q}_i, \tilde{\omega}_i, \omega_d, \dot{\omega}_d) \beta_i + u_i(t) + d_i(t) \quad (7.36)$$

where $F_i(\bar{q}_i, \tilde{\omega}_i, \omega_d, \dot{\omega}_d)$ is given by

$$\begin{aligned} F_i = & -\omega_i^\times L(\omega_i) + L \left[\tilde{\omega}_i^\times R(\bar{q}_i) \omega^d - R(\bar{q}_i) \dot{\omega}^d + \frac{1}{2} \rho_i S(\bar{q}_i) \tilde{\omega}_i \right] \\ & + L \left\{ \frac{1}{2} \sum_{j=1}^n \left[\rho_{ij}^a S(\bar{q}_i) \tilde{\omega}_i - \rho_{ij}^b S(\bar{q}_j(t - T_{ij})) \tilde{\omega}_j(t - T_{ij}) \right] \right\} \end{aligned} \quad (7.37)$$

The adaptive control law for the i th satellite is designed as follows

$$u_i(t) = -c_i q_i - K_i z_{2i} - F_i(\bar{q}_i, \tilde{\omega}_i, \omega_d, \dot{\omega}_d) \hat{\beta}_i - \hat{D}_i \operatorname{sgn}(z_{2i}) \quad (7.38)$$

$$\dot{\hat{\beta}}_i = \Gamma_i [F_i(\bar{q}_i, \tilde{\omega}_i, \omega_d, \dot{\omega}_d)]^T z_{2i} \quad (7.39)$$

$$\dot{\hat{D}}_i = \Phi_i \|z_{2i}\| \quad (7.40)$$

where K_i and Γ_i are positive definite matrices, Φ_i is a positive constant, $\hat{\beta}_i$ and $\hat{D}_i$ are estimates of β_i and D_i , respectively. $\operatorname{sgn}(z_{2i}) = [\operatorname{sgn}(z_{2i}^1) \ \operatorname{sgn}(z_{2i}^2) \ \operatorname{sgn}(z_{2i}^3)]^T$, z_{2i}^j , $j = 1, 2, 3$ is the i th component of z_{2i} .

Choose the candidate of Lyapunov function as

$$V_2 = V_1 + \frac{1}{2} \sum_{i=1}^n z_{2i}^T J_i z_{2i} + \frac{1}{2} \tilde{\beta}_i^T \Gamma_i^{-1} \tilde{\beta}_i + \frac{1}{2} \tilde{D}_i^T \Phi_i^{-1} \tilde{D}_i \quad (7.41)$$

where $\tilde{\beta}_i = \beta_i - \hat{\beta}_i$ and $\tilde{D}_i = D_i - \hat{D}_i$.

The derivative of V_2 is

$$\begin{aligned}
\dot{V}_2 &= \sum_{i=1}^n c_i q_i^T \left[z_{2i} - \rho_i q_i - \sum_{j=1}^n \left(\rho_{ij}^a q_i - \rho_{ij}^b q_j (t - T_{ij}) \right) \right] + \sum_{i=1}^n z_{2i}^T J_i \dot{z}_{2i} + \tilde{\beta}_i^T \Gamma_i^{-1} \dot{\tilde{\beta}}_i + \tilde{D}_i^T \Phi_i^{-1} \dot{\tilde{D}}_i \\
&\leq - \sum_{i=1}^n c_i \rho_i q_i^T q_i - \sum_{i=1}^n \sum_{j=1}^n \varepsilon_{ij}^T \varepsilon_{ij} + \sum_{i=1}^n z_{2i}^T (c_i q_i + F_i \beta_i + u_i + d_i) + \tilde{\beta}_i^T \Gamma_i^{-1} \dot{\tilde{\beta}}_i + \tilde{D}_i^T \Phi_i^{-1} \dot{\tilde{D}}_i \\
&\leq - \sum_{i=1}^n c_i \rho_i q_i^T q_i - \sum_{i=1}^n \sum_{j=1}^n \varepsilon_{ij}^T \varepsilon_{ij} - \sum_{i=1}^n z_{2i}^T K_i z_{2i} \\
&\quad + \sum_{i=1}^n z_i^T \left(F_i \tilde{\beta}_i - \hat{D}_i \operatorname{sgn}(z_{2i}) + d_i \right) + \tilde{\beta}_i^T \Gamma_i^{-1} \dot{\tilde{\beta}}_i + \tilde{D}_i^T \Phi_i^{-1} \dot{\tilde{D}}_i \\
&\leq - \sum_{i=1}^n c_i \rho_i q_i^T q_i - \sum_{i=1}^n \sum_{j=1}^n \varepsilon_{ij}^T \varepsilon_{ij} - \sum_{i=1}^n z_{2i}^T K_i z_{2i} \\
&\quad + \sum_{i=1}^n \tilde{D}_i^T \Phi_i^{-1} \left(\Phi_i \|z_{2i}\| - \dot{\tilde{D}}_i \right) + \sum_{i=1}^n \tilde{\beta}_i^T \Gamma_i^{-1} \left[\Gamma_i F_i^T z_{2i} - \dot{\tilde{\beta}}_i \right]
\end{aligned} \tag{7.42}$$

Substituting (7.39) and (7.40) into the above equation yields

$$\dot{V}_2 = - \sum_{i=1}^n c_i \rho_i q_i^T q_i - \sum_{i=1}^n \sum_{j=1}^n \varepsilon_{ij}^T \varepsilon_{ij} - \sum_{i=1}^n z_{2i}^T K_i z_{2i} \leq 0 \tag{7.43}$$

The following stability results based on this scheme can be derived

Theorem 7.2 Consider attitude synchronization and tracking systems described by (6.11)–(6.13) with application of controller (7.38) and the parameter update laws (7.39) and (7.40). If Assumptions 6.1, 6.6 and 7.1, 7.2 are valid and the control parameters satisfy the constraints in (7.22) and (7.23), then the attitude synchronization and tracking problem in Definition 6.1 is asymptotically solved.

Proof From (7.43), it is established that V_2 is non-increasing. Therefore, it follows that $z_{1i}, z_{2i} \in \mathcal{L}^\infty$, and $\hat{\beta}_i, \hat{D}_i \in \mathcal{L}^\infty$. Thus, $u_i \in \mathcal{L}^\infty$ and $\dot{z}_{1i}, \dot{z}_{2i}$ are all bounded. Integrating $\dot{V}$ gives the results that $z_{1i}, z_{2i} \in \mathcal{L}^2$. Hence, using the corollary of Barbalat's Lemma in (Sastry and Bodson 1989), it follows that $\lim_{t \rightarrow \infty} z_{1i}(t) = 0$ and $\lim_{t \rightarrow \infty} z_{2i}(t) = 0$. Thus, it follows that $\lim_{t \rightarrow \infty} q_i(t) = 0$ and $\lim_{t \rightarrow \infty} \tilde{\omega}_i(t) = 0$ by (7.31). $\square$

Remark 7.8 The smoothing control technique introduced in Sect. 6.4.3 can also be deployed to avoid the chattering of control input in (7.38).

Theorem 7.2 is a general conclusion which always takes effect as long as the inter-satellite communication time delays satisfy Assumption 7.2, which means that if Assumption 7.2 is valid, then controller (7.38) is competent, whether the time delays are large or small, and whether the variables of the entire system change quickly or slowly. In fact, if the time delays in the communication links are small enough, or the attitude states of the whole formation are so slowly time varying that

the effects of the delays can be ignored, then some extended results can be obtained as presented in the following.

Corollary 7.1 *Consider attitude synchronization and tracking systems described by (6.11)–(6.13) with application of the following control law and the parameter updating laws.*

$$u_i(t) = -c_i q_i - K_i z_{2i} - F_i(\bar{q}_i, \tilde{\omega}_i, \omega_d, \dot{\omega}_d) \hat{\beta}_i - \hat{D}_i \operatorname{sgn}(z_{2i}) \quad (7.44)$$

$$\dot{\hat{\beta}}_i = \Gamma_i [F_i(\bar{q}_i, \tilde{\omega}_i, \omega_d, \dot{\omega}_d)]^T z_{2i} \quad (7.45)$$

$$\dot{\hat{D}}_i = \Phi_i \|z_{2i}\| \quad (7.46)$$

with

$$z_{2i} = \omega_i + \rho_i^p q_i + \sum_{j=1}^n \left(\rho_{ij}^a q_i - \rho_{ij}^b q_j \right) \quad (7.47)$$

$$\begin{aligned} F_i = & -\omega_i^\times L(\omega_i) + L \left[\tilde{\omega}_i^\times R(\bar{q}_i) \omega^d - R(\bar{q}_i) \dot{\omega}^d + \frac{1}{2} \rho_i S(\bar{q}_i) \tilde{\omega}_i \right] \\ & + L \left\{ \frac{1}{2} \sum_{j=1}^n \left[\rho_{ij}^a S(\bar{q}_i) \tilde{\omega}_i - \rho_{ij}^b S(\bar{q}_j) \tilde{\omega}_j \right] \right\} \end{aligned} \quad (7.48)$$

If Assumptions 6.1, 6.6, and 7.1, 7.2 are valid and the control parameters satisfy the constraints in (7.22) and

$$\rho_{ij}^b \leq \sqrt{4\gamma_{ij} \left(c_i \rho_{ij}^a - \gamma_{ij} \right) / c_i^2} \quad (7.49)$$

then the attitude synchronization and tracking problem in Definition 6.1 is asymptotically solved.

Following the stability proof of Theorem 7.2, the stability proof of corollary can be obtained straightforwardly.

Remark 7.9 The smoothing control technique introduced in Sect. 6.4.3 can also be deployed to avoid the chattering of control input (7.44).

7.4 Numerical Results

Simulation results are presented in this section to illustrate the performance and stability characteristics of the proposed decentralized adaptive control laws. Two scenarios with four satellites are considered in the simulations. In the first scenario,

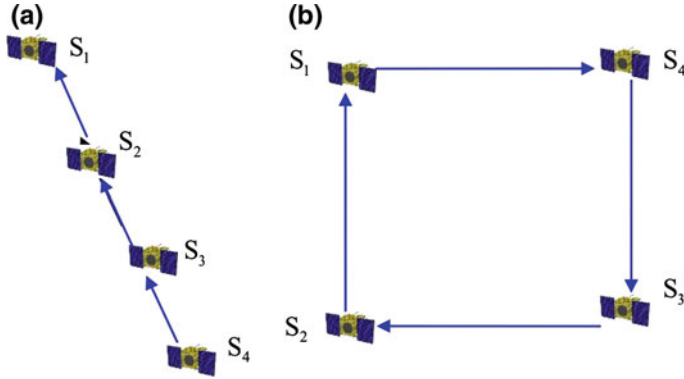


Fig. 7.1 Inter-satellite directed communication topology for satellite attitude synchronization. **a** Line topology for in-track/in-plane formation. **b** Ring topology for circular-like formation

the four satellites communicate via a directed line topology as shown in Fig. 7.1a, while a ring topology, shown in Fig. 7.1b, is used in the second scenario. Note that although both communication graphs have a spanning tree, the proposed control law is valid even when there is no spanning tree. The corresponding weighted Laplacian matrices for the topologies are

$$L_a = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad L_b = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix} \quad (7.50)$$

The following two subsections present simulation results of decentralized adaptive sliding mode control and adaptive backstepping control for attitude synchronization and tracking, respectively.

7.4.1 Decentralized Adaptive Sliding Mode Control

The simulation parameters, such as the actual and nominal inertia matrices of the satellites, the external disturbances, the initial attitude tracking errors, the initial angular velocity errors, and the desired attitude and angular velocity, are the same as in Sect. 6.6.1. Simulation results of adaptive sliding mode control for Scenario 1 and Scenario 2 are given as follows.

Scenario 1: line communication topology

The controller parameters in the first scenario are chosen with $C = I_3$, $b_i = 1$, $\phi_i = 0.1$, $K_i = 0.01I_3$ and the parameters of the adaptation law (7.9) are chosen with $\bar{\kappa}_{i,0} = 0.1$, $\bar{\kappa}_{i,1} = \bar{\kappa}_{i,2} = 0.2$, for $i = 1, \dots, 4$ and $\hat{c}_{i,0}^0 = \hat{c}_{i,1}^0 = \hat{c}_{i,2}^0 = 0$, where

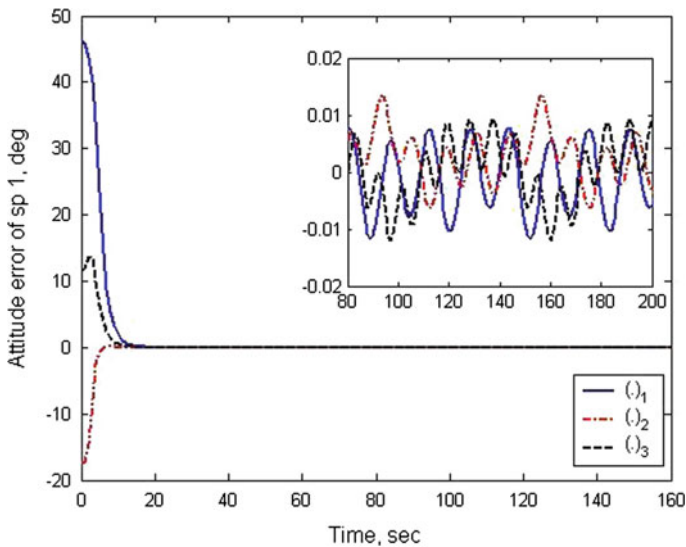


Fig. 7.2 Attitude tracking error of the 1st satellite under decentralized robust adaptive control with directed communication topology

$\hat{c}_{i,0}^0, \hat{c}_{i,1}^0, \hat{c}_{i,2}^0$ are the initial values of $\hat{c}_{i,0}, \hat{c}_{i,1}, \hat{c}_{i,2}$, respectively. The control parameters of inter-satellite synchronization are chosen as $a_{12} = 1, a_{23} = 1, a_{34} = 1$ and all other $a_{ij} = 0$.

Figures 7.2 and 7.3 show the respective attitude errors and control torques of the 1st satellite using the control law (7.10), respectively. For ease of interpretation, attitude errors are expressed in Euler angles converted from unit quaternion. Attitude errors and control torques of the other satellites are similar to those of the 1st satellite, and are not plotted here due to space constraint. Figures 7.4 and 7.5 show relative attitude errors of the 2nd with respect to the 1st satellite and the 1st with respect to the 4th satellite, respectively. Relative attitude errors between other pairs of satellites are similar to those of the above two. For comparison, Figs. 7.7, 7.8, and 7.9 show attitude errors, control torques, and relative attitude errors when there is no coupling between neighbors [i.e., $a_{ij} = 0, i, j = 1, 2, 3, 4$ in (7.10)]. As observed from simulation results, relative attitude errors between satellites, especially the steady-state values, in the case with coupling between neighbors are smaller as compared to the case without coupling. Note that the magnitudes of steady-state control torques for the two cases are almost identical, as can be seen in Figs. 7.2 and 7.7. The adaptive parameter $\hat{\beta}_1$ defined in (7.11) with the control law (7.10) is bounded as shown in Fig. 7.6, thus the efficacy of the proposed adaptation law (7.9) is verified.

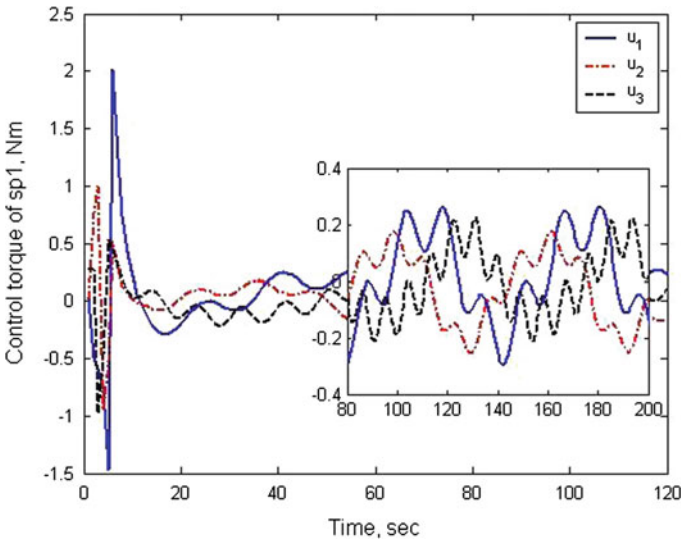


Fig. 7.3 Control torque of the 1st satellite under decentralized robust adaptive control with directed communication topology

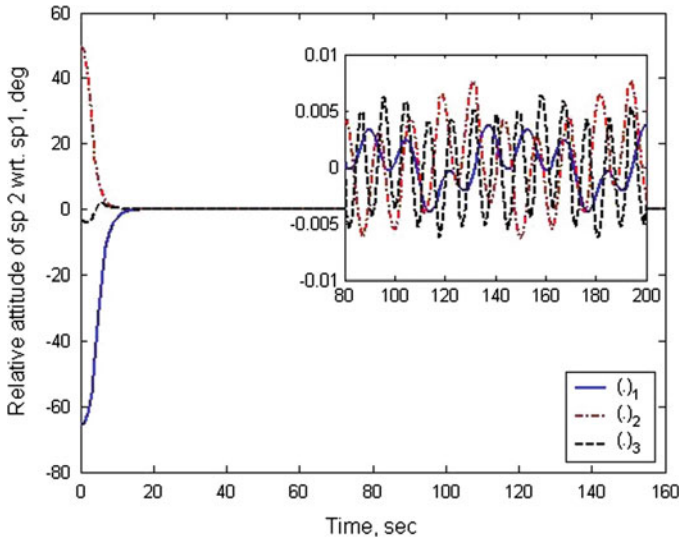


Fig. 7.4 Relative attitude error between the 1st and the 2nd satellites under decentralized robust adaptive control with directed communication topology

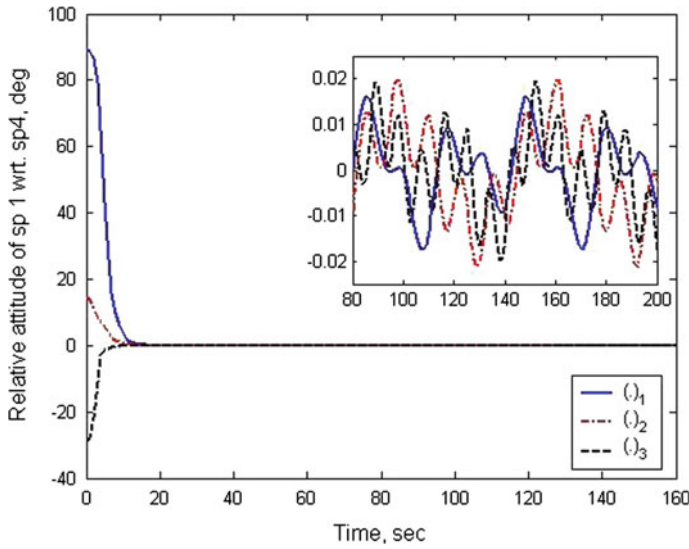
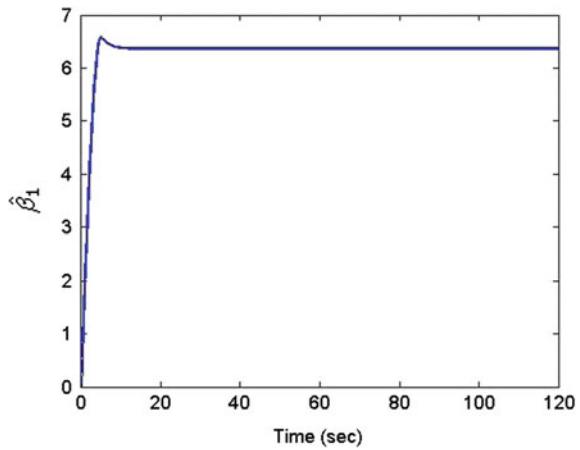


Fig. 7.5 Relative attitude error between the 4th and the 1st satellites under decentralized robust adaptive control with directed communication topology

Fig. 7.6 Adaptive parameter $\hat{\beta}_1$ of the 1st satellite under decentralized robust adaptive control with directed communication topology



Scenario 2: ring communication topology

All the controller parameters in the second scenario are identical to those in the first scenario except that $a_{41} = 1$ in the second scenario. Figures 7.10 and 7.11 show relative attitude errors between the 1st and 2nd satellites and between 4th and 1st satellites, respectively. As observed from Figs. 7.4, 7.5, 7.10 and 7.11, relative attitude errors in the ring topology are slightly smaller than those in the line topology, because additional coupling between 4th to 1st satellites is introduced in

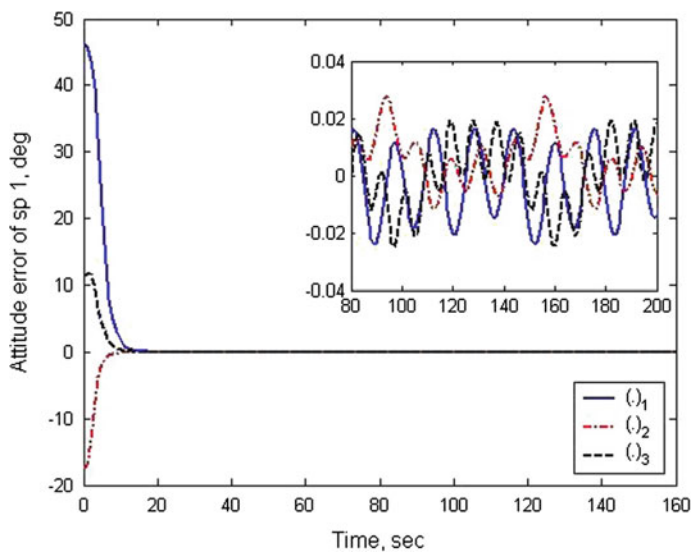


Fig. 7.7 Attitude tracking error of the 1st satellite without coupling between neighbors under decentralized robust adaptive control with directed communication topology

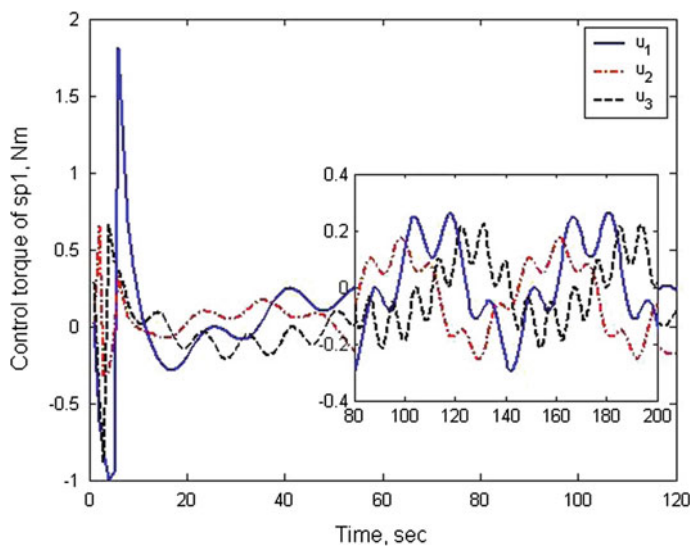


Fig. 7.8 Control torque of the 1st satellite without coupling between neighbors under decentralized robust adaptive control with directed communication topology

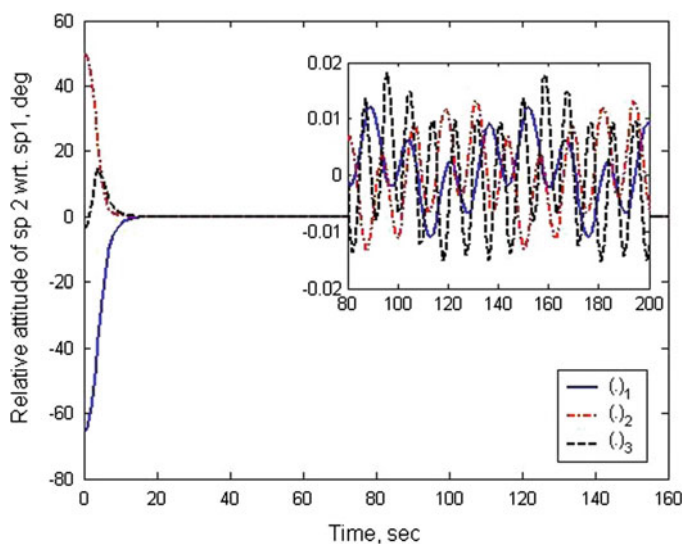


Fig. 7.9 Relative attitude error between the 1st and the 2nd satellites without coupling between neighbors under decentralized robust adaptive control with directed communication topology

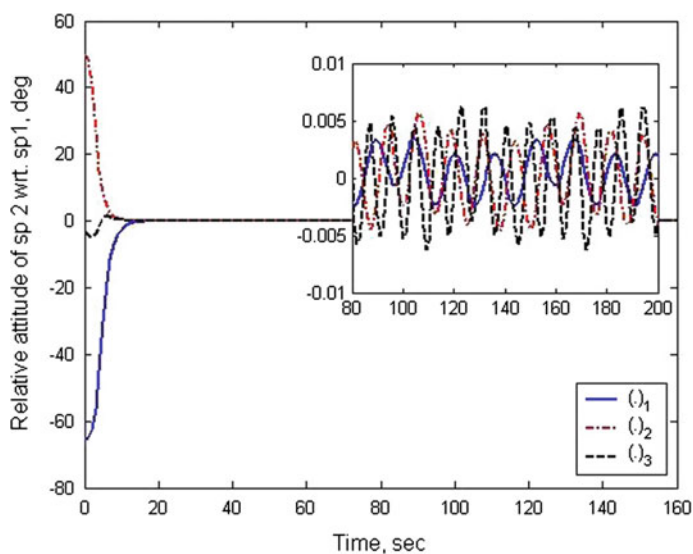


Fig. 7.10 Relative attitude error between the 1st and the 2nd satellites under decentralized robust adaptive control with the unidirectional ring topology

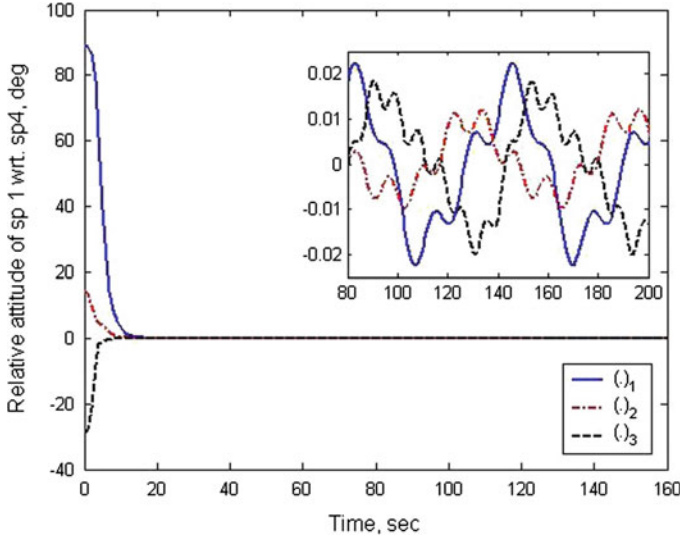


Fig. 7.11 Relative attitude error between the 4th and the 1st satellites under decentralized robust adaptive control with the unidirectional ring topology

the ring topology. Thus, the simulation results validate the effectiveness of the proposed control law under a unidirectional ring communication topology.

7.4.2 Decentralized Adaptive Backstepping Control

The inter-satellite communication topology used in this example is the ring topology for circular-like formation as in Fig. 7.1b.

The actual inertia matrices of the satellites and the external disturbances are the same as in Sect. 6.6.1. The desired angular velocities of the satellites are $\omega_i^d(t) = [0, 0.2, 0]^T$ rad/s, and the initial desired attitudes are $\bar{\eta}_i^d(0) = [1 \ 0 \ 0 \ 0]^T$, where $i = 1, 2, 3, 4$. The initial angular velocities of the satellites are chosen as zero, and the initial attitude tracking errors are chosen as

$$\begin{aligned} \bar{q}_1(0) &= [0.9849 \ 0.1 \ 0.1 \ -0.1]^T, \quad \bar{q}_2(0) = [0.9763 \ 0.1 \ 0.15 \ -0.12]^T, \\ \bar{q}_3(0) &= [0.9605 \ 0.15 \ -0.15 \ 0.18]^T, \quad \bar{q}_4(0) = [0.9750 \ 0.10 \ -0.15 \ 0.13]^T. \end{aligned} \quad (7.51)$$

In order to validate the robustness against model uncertainties of the proposed control laws, it is assumed that the inertia matrices of the satellites are known as

$$\begin{aligned}
J_1 &= \text{diag}([40, 34, 30]^T) \text{ kg m}^2 \\
J_2 &= \text{diag}([44, 38, 30]^T) \text{ kg m}^2 \\
J_3 &= \text{diag}([36, 30, 34]^T) \text{ kg m}^2 \\
J_4 &= \text{diag}([36, 40, 30]^T) \text{ kg m}^2
\end{aligned} \tag{7.52}$$

which are set as the initial values of the adaptation laws (7.39).

For comparison, two cases are considered. In this first case, it is assumed inter-satellite coordination is not available, i.e., all ρ_{ij}^a and ρ_{ij}^b are equal to zero. In the second case, the inter-satellite coordination corresponding to the communication topology in Fig. 7.1b is used.

A. Attitude synchronization without Inter-satellite coordination

The control law parameters for the first case are chosen as $\Gamma_i = \text{diag}([100 \ 100 \ 100, 0.0025 \ 0.0025 \ 0.0025]^T)$, $c_i = 0.1$, $K_i = 5I_3$, $k_{2,i} = 0.3$, $\rho_i = I_3$, $\rho_{ij}^a = 0$, $\rho_{ij}^b = 0$, $\Phi_i = 0.2$ and $\phi_i = 0.01$, where $i = 1, 2, 3, 4$, $j = 1, 2, 3, 4$, $j \neq i$.

The response of the closed-loop system of the 1st satellite for the first case is shown in Figs. 7.12, 7.13, and 7.14. For ease of interpretation, attitude errors are expressed by Euler angles converted from unit quaternion. It is observed that attitude tracking of the 1st satellite is achieved despite the presence of external disturbances and large model uncertainties. The parameter estimations of inertia matrix are bounded, as shown in Fig. 7.14. The responses of the closed-loop systems of other satellites are similar and are not plotted here due to space constraint.

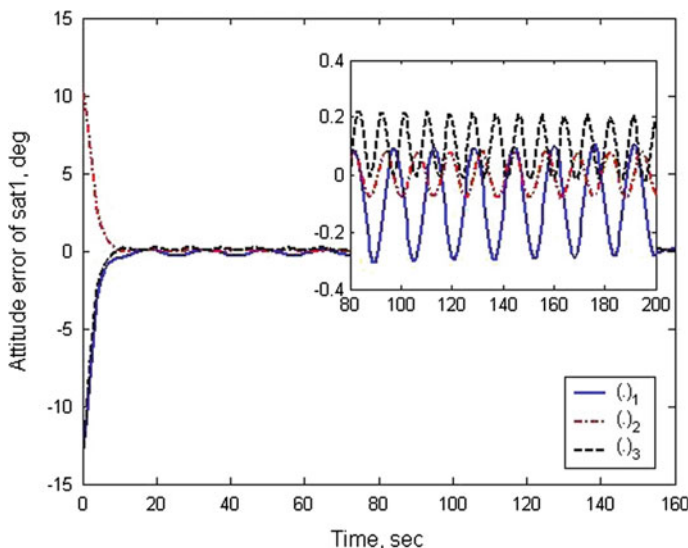


Fig. 7.12 Attitude tracking error of the 1st satellite without coupling between neighbors under decentralized adaptive backstepping control with the unidirectional ring topology

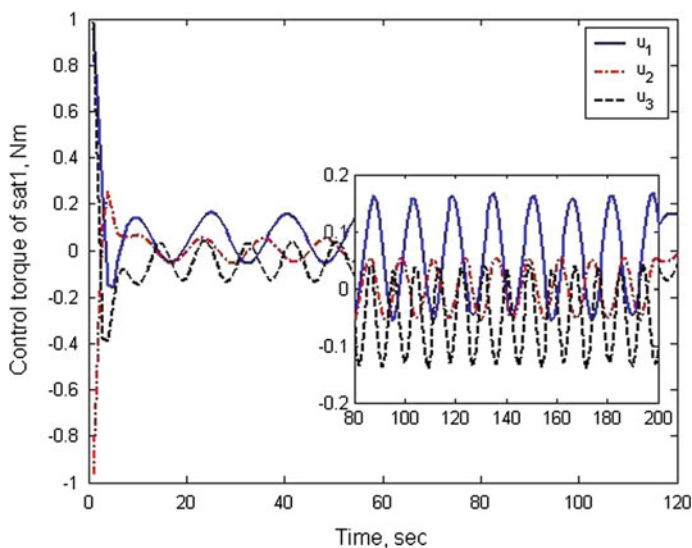


Fig. 7.13 Control torque of the 1st satellite without coupling between neighbors under decentralized adaptive backstepping control with the unidirectional ring topology

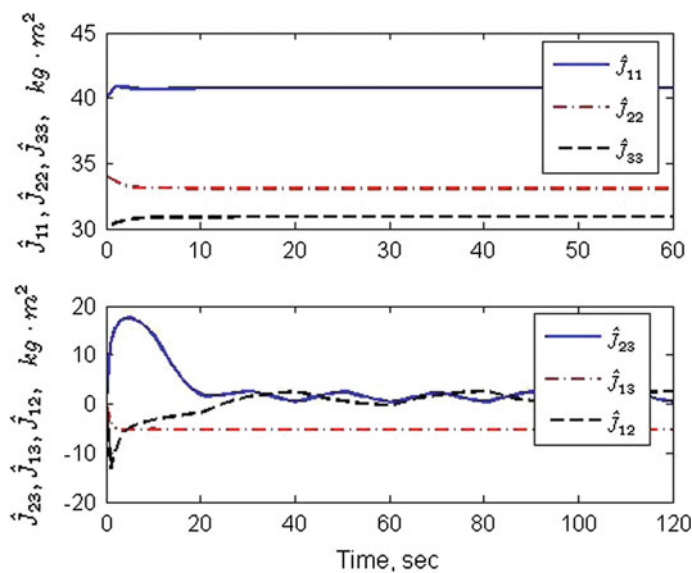


Fig. 7.14 Inertia matrix estimation of the 1st satellite without coupling between neighbors under decentralized adaptive backstepping control with the unidirectional ring topology

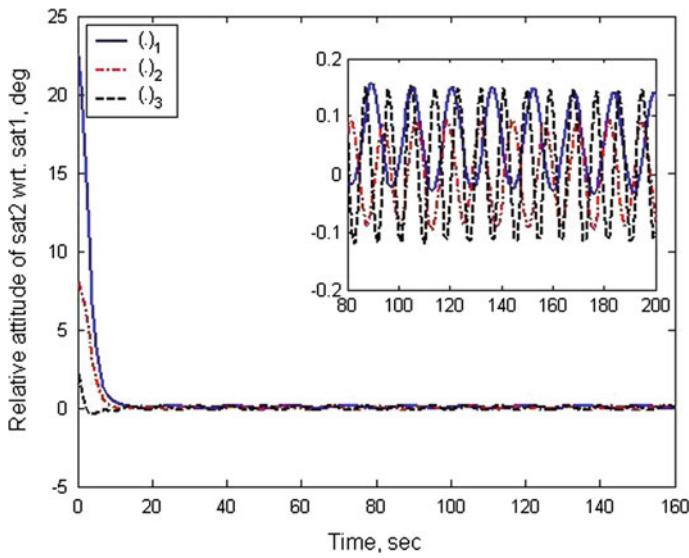


Fig. 7.15 Relative attitude errors of the 2nd with respect to the 1st satellite without coupling between neighbors under adaptive backstepping control with the unidirectional ring topology

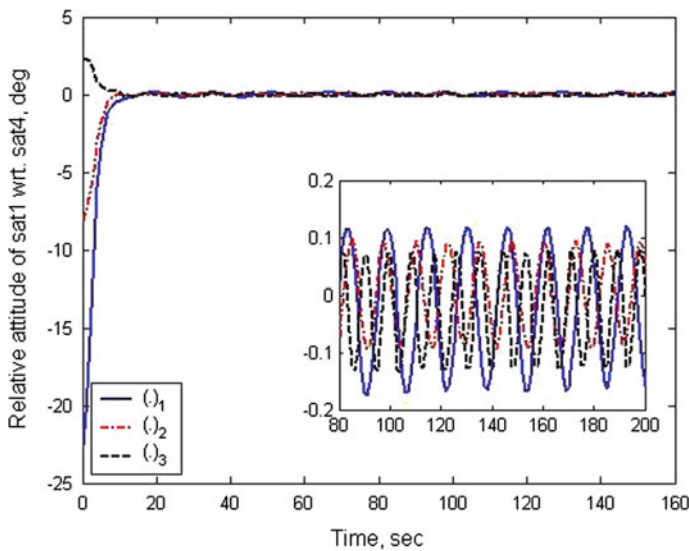


Fig. 7.16 Relative attitude errors of the 1st with respect to the 4th satellite without coupling between neighbors under decentralized adaptive backstepping control with the unidirectional ring topology

The relative attitudes of the 2nd with respect to the 1st satellite and the 1st with respect to the 4th satellite are plotted in Figs. 7.15 and 7.16, respectively.

B. Attitude Synchronization with Inter-Satellite Communication

The control parameters of inter-satellite synchronization for the second case are chosen as $\rho_{12}^a = 1.2$, $\rho_{12}^b = 1$, $\rho_{23}^a = 1.2$, $\rho_{23}^b = 1$, $\rho_{34}^a = 1.2$, $\rho_{34}^b = 1$, $\rho_{41}^a = 1.2$, $\rho_{41}^b = 1$ and all other ρ_{ij}^a, ρ_{ij}^b are zero. All the inter-satellite communication delays are assumed to be identical and equal to $T_{ij} = 1 + 0.1 \sin(0.2t)$ seconds. The assumed communication delay comprises constant time delay 1 s and time-varying delay $0.1 \sin(0.2t)$ second. In order to validate the effectiveness of the proposed control law in the presence of communication delay, the assumed time delay is larger than the case in practice. Other control parameters are identical with the first case.

Simulation results of the second case are shown in Figs. 7.17, 7.18, 7.19, 7.20. Figure 7.17 shows attitude tracking error of the 1st satellite. Attitude errors of the other satellites are similar to that of the 1st satellite. Figures 7.18 and 7.19 show the attitude tracking errors of the 2nd with respect to the 1st satellite and the 1st with respect to the 4th satellite. Relative attitude errors between other pairs of satellites are similar to those of the above two. As observed from Figs. 7.15, 7.16 and 7.18 and 7.19, relative attitude errors among satellites are reduced, especially steady-state relative attitude errors. Although steady-state relative attitude errors are greatly reduced in the second case, the control torques in the steady-state process for the two cases are almost identical, as can be shown in Figs. 7.13 and 7.20. Thus, simulation results validate the effectiveness of the proposed decentralized adaptive backstepping control law for satellite attitude tracking synchronization problem with communication delay.

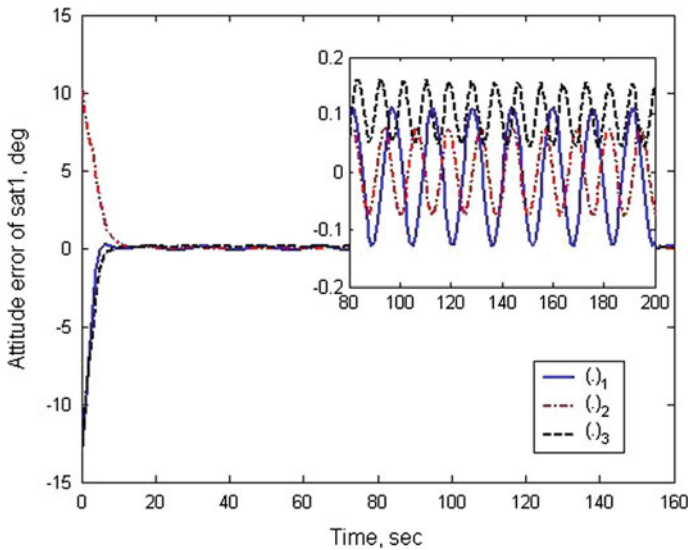


Fig. 7.17 Attitude tracking error of the 1st satellite under decentralized adaptive backstepping control with the unidirectional ring topology

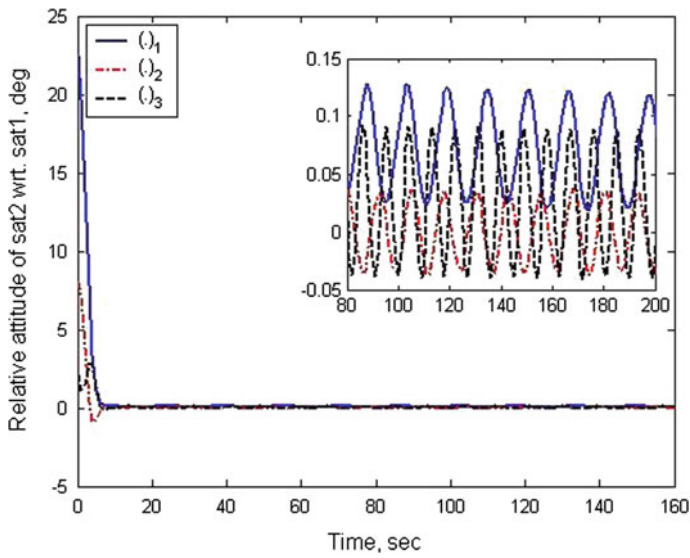


Fig. 7.18 Relative attitude error from the 1st to the 2nd satellite under decentralized adaptive backstepping control with the unidirectional ring topology

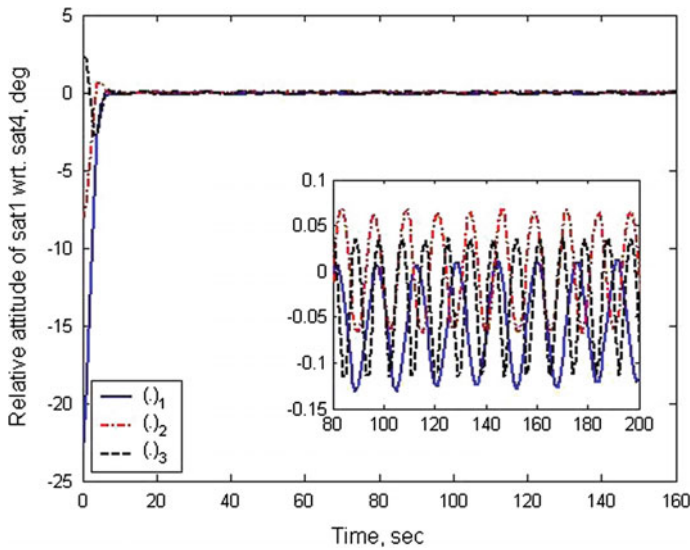


Fig. 7.19 Relative attitude error from the 4th to the 1st satellite under decentralized adaptive backstepping control with the unidirectional ring topology

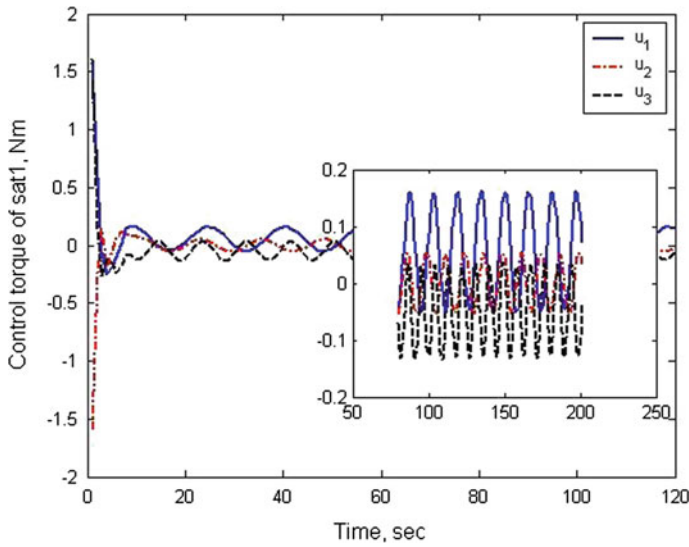


Fig. 7.20 Control torque of the 1st satellite under decentralized adaptive backstepping control with the unidirectional ring topology

7.5 Conclusions

In this chapter, the attitude synchronization and tracking problem in satellite formation under a general directed communication topology is addressed using unit quaternion parameterization. Firstly, a decentralized adaptive sliding mode control law is designed by introducing appropriate multi-satellite sliding mode vector, which includes attitude error and angular velocity error of individual satellite, as well as relative attitude errors and relative angular velocity errors between satellites. An adaptive mechanism to estimate the model uncertainty and external disturbance bound was presented. Convergence of the tracking errors was established using graph theoretic formulation. Then, an adaptive backstepping control law is proposed for a general directed communication topology with communication time delay. Numerical simulations are performed to validate the robust performance of the proposed control laws in the presence of model uncertainties, disturbances, and communication delay (for the second control law). Simulation results demonstrate that each individual satellite converges to the desired attitude and angular velocity with acceptable control magnitude. Furthermore, relative attitude errors between satellites, especially the steady-state values, are reduced by including the relative attitude errors and relative angular velocity errors in the proposed control laws.

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Chapter 8

Conclusions

8.1 Summary

The primary motivation for satellite formation flying is that multiple small satellites can work together in a group to accomplish the mission of a larger, usually more expensive, satellite. Operating multiple satellites in the formation offers benefits of (i) flexibility because the operator can reconfigure and reposition the individual satellites to perform different tasks, (ii) graceful degradation of performance in the event of single satellite failure which will not result in total mission failure due to higher redundancy, and (iii) a more cost effective solution including cheaper and simpler bus designs and faster build times. These attributes have engendered satellite formation flying for many interesting missions in the areas of astronomy, communications, meteorology, and environmental monitoring.

The control of satellite formation flying can conceptually be realized in two ways:

- **Ground-based or Space-based Leader-Follower centralized control**—In ground-based control, the respective measurements are sent by each satellite on the formation to a ground control center that will generate control commands to enable these satellites to maneuver into the appropriate position in the formation. This approach is adequate for formations with separation between spacecraft on the order of kilometers without requiring dynamic adjustment of their orbits. The interval between adjustment maneuvers is usually done over months. Similarly, the leader-follower approach is another form of centralized control solution in which the leader is a single point of failure for the entire formation.
- **Autonomous decentralized control**—In autonomous flying formations, the individual satellite measurements are broadcasted between spacecraft, allowing them to compute their relative positions in the formation and use the Attitude and Orbit Control Systems (AOCS) to maneuver into the desired position and attitude orientation. This approach is necessary for close satellite formations that require frequent adjustment of their relative positions and orientation. This

approach is technically more difficult and challenging since unforeseen disturbances and collision risks can potentially lead to loss of control or destruction of the spacecraft.

Indeed, advanced satellite formation flying concepts have become more realizable mainly due to the development of autonomous decentralized control capability. With an onboard computer, satellites will be able to autonomously position and orientate themselves into desired formation without requiring ground control intervention to continuously adjust each satellite to maintain formation. As such, satellites are able to arrive at and maintain formations with faster response time and have the ability to change the formation to meet mission requirements even in the presence of environmental disturbances and system uncertainties. Also, satellites may be launched from different launchers and maneuvered to rendezvous on specific orbit. This is true particularly for satellite cluster formation with satellites operating in a dense (relatively tightly spaced) arrangement. The resulting formations are best suited for high-resolution interferometry and Digital Elevation Measurement. The TechSat-21 is an example of satellite cluster formation flying.

This book has made further contribution in this areas of centralized and decentralized control for satellite formation flying. We reviewed common relative satellite motion models which are needed for satellite formation control design and proposed a novel dynamic model for satellite relative motion for precise satellite formation design, guidance, and control. Unlike earlier work, we obtain an exact J_2 nonlinear relative model which ensures high satellite orbit fidelity and accuracy required for precise long-duration satellite formation control and maintenance. A simpler first order J_2 linear relative model is also derived with reduced computational requirement and is suitable for periodic short-term formation maneuver and keeping.

Passive and periodic satellite formation (Sabol et al. 2001) is another interesting problem discussed in this book. Such formation is attractive as it eliminates, or at least minimizes, the thruster propellant needed for maintenance and thus improves the useable duration of the mission. We extend earlier results which ignore J_2 gravitational perturbations on the satellite orbits. Under this condition, it is shown that there are two classes of passive and periodic relative trajectories: (i) one with all satellites flying in the critical inclination and (ii) all satellites flying in the same regressing orbital plane. The respective exact conditions for these two trajectories are established and proved.

The problem of precise decentralized satellite formation control using low-thrust electric propulsion system is also addressed. Recent advances in electric propulsion system which promises precise thrust output and consume less propellant have made it an attractive solution (Polzin 2011). Furthermore, using the exact J_2 nonlinear relative model developed earlier in this book, a nonlinear optimal control problem is formulated to design fuel-optimal trajectories for satellite formation maneuver using low-thrust continuous propulsion. The Legendre pseudospectral method is applied to solve the optimization problem with desired terminal

formation geometry requirements and collision avoidance constraints using low-thrust propulsion system.

Formation keeping strategy is necessary to ensure that desired satellite formation flying is maintained in the presence of various environmental perturbations such as gravitational perturbation and atmospheric drag. Two methods of optimal control-based satellite formation keeping strategies are presented. The first approach develops a real-time fuel-optimal control law for a leader-follower architecture which maintains the formation despite various orbital perturbations. The second solution adopts a decentralized formation control approach using only local relative measurement. This latter methodology is completely scalable as it does not require full communication among all the formation satellites.

Attitude synchronization among satellites is beneficial for practical uses such as satellite surveillance, pointing control, formation flying, and space-based interferometry (Li and Kumar 2012).

This book discusses decentralized control for attitude synchronization using both undirected, i.e., bidirectional, and directional inter-satellite communication topology. With undirected communication topology, a coordinated attitude control law without velocity measurement is derived to solve the attitude tracking synchronization problem. In practice, the inter-satellite communication topology may be restricted to be directed, such as in unidirectional satellite laser communication system. Hence, controller synthesis solutions were also developed which are robust against inertia matrix uncertainties, external disturbances and communication time latency.

8.2 Trends and Challenges

In recent years, there have been a growing number of small satellite missions, in various stages of development, for formation flying missions. In particular, small nano-satellites, weighing between 1 and 8 kg, are particularly attractive for formation flying missions because of their modular nature and low cost. These small satellites can be configured to serve as distributed instruments for atmospheric sampling, construct a large distributed antenna platform, or constitute a large distributed aperture for imaging and surveillance applications.

In Bandyopadhyay et al. (2015), a survey review of state-of-the-art satellite formation flying missions, using multiple small satellites, was presented. The thirty-nine missions were also categorized based on their mission type, mission status, and number of satellites. It was concluded that Earth science-related missions are the most popular. Six missions target to demonstrate formation flying capability using two to three small satellites while another 2 missions plan to use four or more small satellites. The rest of the multi-satellite missions require a constellation.

In essence, the primary technical challenges for satellite formation control design and implementation includes the following:

- Controlling the nonlinear dynamics in satellite formation flying in the presence of environmental disturbances with autonomous control algorithms.
- Balancing the requirements between satellite maneuver for satellite formation keeping and fuel consumption
- Developing sub-millimeter accuracy sensors to determine the relative position of satellites and maturing actuator technology to enhance orbital position control for small satellites.
- Overcoming robustness challenges for inter-satellite communication.

In this book, we have addressed the first two technologies but not the navigation, sensors, actuators and communication hardware required for formation flying missions. Some studies have suggested the use of Global Navigation Satellite System (GNSS) for relative navigation of near-Earth formation flying missions. However, GNSS is not suitable for precision relative navigation in formation flying beyond low Earth orbits. Also, full collaboration and communication architecture among the formation satellites is not preferable for some flying missions due to high operational cost and technology immaturity. These remain interesting and active areas of research and development.

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Appendix A

Algebraic Graph Theory

The necessary results from algebraic graph theory (Merris 1994; Chung 1997) are introduced in this section to address the decentralized formation keeping and cooperative attitude synchronization problems. A directed communication topology can be described by directed graph. A directed graph G_n consists of a finite set of vertices, denoted V , and a set of arcs $A \subset V^2$, where $a = (\bar{\alpha}, \bar{\beta}) \in A$ and $\bar{\alpha}, \bar{\beta} \in V$. The arc $(\bar{\alpha}, \bar{\beta})$ denotes that vertex $\bar{\beta}$ can obtain the information of vertex $\bar{\alpha}$. In decentralized control problem for satellite formation, the arc (α, β) denotes that satellite β can measure relative state of satellite α with respect to satellite β . In attitude synchronization application, the arc $(\bar{\alpha}, \bar{\beta})$ denotes that satellite $\bar{\beta}$ can obtain attitude information of satellite $\bar{\alpha}$. It is assumed that the graph has no self-loops, meaning that $(\bar{\alpha}, \bar{\beta}) \in A$ implies $\bar{\alpha} \neq \bar{\beta}$. If every possible arc exists, the graph is said to be complete.

A path on G_n of length N from $\bar{\alpha}_1$ to $\bar{\alpha}_{N+1}$ is an ordered set of distinct vertices $\{\bar{\alpha}_1, \dots, \bar{\alpha}_{N+1}\}$ such that $(\bar{\alpha}_{i-1}, \bar{\alpha}_i) \in A$ for all $i \in [2, \dots, N+1]$. A graph in which a path exists from any vertex to any other vertex is said to be strongly connected. A spanning tree in G_n is a graph in which exists a vertex $\bar{\alpha}$ such that there is a directed path from vertex α to every other vertex in G_n .

The adjacency matrix of G_n denoted A is a square matrix of size n with entries

$$\begin{cases} a_{i,j} > 0 & \text{if } (\bar{\alpha}_j, \bar{\alpha}_i) \in A \\ a_{i,j} = 0 & \text{otherwise} \end{cases} \quad (\bar{\alpha}_i, \bar{\alpha}_j \in \mathbf{V}) \quad (\text{A.1})$$

where the non-negative $a_{i,j}$ is subsequently chosen to be the control weight parameter between the i th and j th satellite. For decentralized formation keeping, $w_{ij} a_{i,j}$ can be used to balance fuel consumption among satellites. Note that $a_{i,i} = 0$ from (A.1).

The in-degree matrix of G_n is the diagonal matrix D with diagonal entries

$$d_{i,i} = \sum_{j=1, j \neq i}^n a_{i,j}, \quad i = 1, \dots, n \quad (\text{A.2})$$

Following Merris (1994), the Laplacian $L \in \mathbb{R}^{n \times n}$ of the graph G_n is defined as follows:

$$L = D - A \quad (\text{A.3})$$

Note that a graph with the property that for any $(\bar{\alpha}, \bar{\beta}) \in A$, the arc $(\bar{\beta}, \bar{\alpha}) \in A$ as well is said to be undirected. In the satellite formation application, this corresponds to having bidirectional relative measurement or communication. It is valid to assume $a_{i,j} = a_{j,i}$ in the case of the undirected communication\ sensing topology. Under this assumption, the Laplacian L is a symmetrical matrix, which simplifies the stability analysis of cooperative control system. However, in the case of directed communication\ sensing topology, L is generally not symmetric because $a_{i,j} = a_{j,i}$ does not hold.

The following results are used in Chaps. 6 and 7 to derive stability proof for the proposed controllers design.

Proposition A.1 (Merris 1994) *Zero is an eigenvalue of L . The associated eigenvector is all the ones vector 1^T .*

Proposition A.2 (Merris 1994) *For a digraph with N vertices, all the eigenvalues of L have nonnegative real part less than or equal to $2(N - 1)$ (use Gershgorin's theorem). Moreover, except for the eigenvalue zero, the real part of all other eigenvalues is positive.*

Proposition A.3 (Merris 1994) *If G_n is undirected, then all the eigenvalues of L are real. Moreover, the least nonzero eigenvalue λ_2 of L grows monotonically with the number of arcs. More precisely, adding edges never decreases λ_2 .*

Proposition A.4 (Ren et al. 2005) *If and only if G_n has a spanning tree, zero is an eigenvalue of algebraic multiplicity one for the Laplacian L . The associated eigenvector is all the ones vector 1_n^T . The associated left eigenvector is $[v_1, v_2, \dots, v_n]$, where $\sum_{i=1}^n v_i = 1$, and $v_i \geq 0$, $i = 1, 2, \dots, n$.*

Appendix B

Optimal Guaranteed Cost Control

This section introduces a control method called the optimal guaranteed cost control, which is applied to a linear norm-bound uncertain system; details are given in Li (1985).

Consider a class of linear uncertain systems described by the following state-space equation:

$$\dot{\bar{x}}'(\theta) = (\bar{A} + \Delta\bar{A})\bar{x}(\theta) + (\bar{B} + \Delta\bar{B})\bar{u}(\theta), \quad \bar{x}(0) = \bar{x}_0 \quad (\text{A.4})$$

where θ is the free variable, $\bar{x}'(\theta)$ is the derivative of the state variable $\bar{x}$ with respect to θ , $\bar{A}$ and $\bar{B}$ are known constant real matrices of appropriate dimensions, $\Delta\bar{A}$, and $\Delta\bar{B}$ are matrix-valued functions representing time-varying parameter uncertainties in the system model. The parameter uncertainties considered here are assumed to be norm-bounded and of the form:

$$[\Delta\bar{A} \quad \Delta\bar{B}] = DF(\theta)[E_1 \quad E_2] \quad (\text{A.5})$$

where D , E_1 and E_2 are known constant real matrices of appropriate dimensions, which represent the structure of uncertainties. Suppose the uncertain matrix $F(\theta)$ is of the following block diagonal form:

$$F(\theta) = \text{diag}\{F_1(\theta), F_2(\theta), \dots, F_l(\theta)\} \quad (\text{A.6})$$

where $F_k(\theta) \in R^{i_k \times j_k}$ and satisfies

$$F_k^T(\theta)F_k(\theta) \leq I_{j_k \times j_k}, \quad k = 1, 2, \dots, l. \quad (\text{A.7})$$

where $I_{j_k \times j_k}$ denotes the identity matrix of dimension j_k . For

$$\epsilon = [\epsilon_1, \epsilon_2, \dots, \epsilon_l], \quad \epsilon_k > 0, \quad k = 1, 2, \dots, l \quad (\text{A.8})$$

Define

$$\tilde{M} = \text{diag}\{\varepsilon_1 I_{i_1 \times i_1}, \varepsilon_2 I_{i_2 \times i_2}, \dots, \varepsilon_l I_{i_l \times i_l}\} \quad (\text{A.9})$$

$$\tilde{N} = \text{diag}\{\varepsilon_1^{-1} I_{j_1 \times j_1}, \varepsilon_2^{-1} I_{j_2 \times j_2}, \dots, \varepsilon_l^{-1} I_{j_l \times j_l}\} \quad (\text{A.10})$$

Then

$$\text{DF}(\theta)[E_1 \ E_2] = D\tilde{M}F(\theta)[\tilde{N}E_1 \ \tilde{N}E_2] \quad (\text{A.11})$$

Associated with the system in (A.4) is the following cost function, where $\bar{Q}$ and $\bar{R}$ are given positive definite symmetric matrices.

$$J = \int_0^\infty [\bar{x}^T(\theta)\bar{Q}\bar{x}(\theta) + \bar{u}^T(\theta)\bar{R}\bar{u}(\theta)] \, d\theta \quad (\text{A.12})$$

Definition A.1 Consider the uncertain system in (A.4), if there exists a control law $\bar{u}^*(\theta)$ and a positive scalar J^* such that for all admissible uncertainties, the closed-loop system is stable and the closed-loop value of the cost function in (A.12) satisfies $J \leq J^*$, then J^* is said to be a guaranteed cost and $\bar{u}^*(\theta)$ is said to be a guaranteed cost control law for the uncertain system in (A.4).

The objective of this section is to design a state-feedback guaranteed cost control law $\bar{u}(\theta) = K\bar{x}(\theta)$ for the uncertain system in (A.4) .

Theorem A.1 (Li 1985) *Given the system described (A.4) and the associated cost function in (A.12), if the following optimization problem*

$$\min_{\varepsilon, W, X, M} \text{Trace}(M) \quad (\text{A.13})$$

subject to

$$\begin{bmatrix} \bar{A}X + \bar{B}W + (\bar{A}X + \bar{B}W)^T & (E_1X + E_1W)^T & X & W^T & D\tilde{M} \\ E_1X + E_1W & -\tilde{N}^{-1} & 0 & 0 & 0 \\ X & 0 & -\bar{Q}^{-1} & 0 & 0 \\ W & 0 & 0 & -\bar{R}^{-1} & 0 \\ \tilde{M}D^T & 0 & 0 & 0 & -\tilde{M} \end{bmatrix} < 0 \quad (\text{A.14})$$

$$\begin{bmatrix} M & I \\ I & X \end{bmatrix} > 0 \quad (\text{A.15})$$

has a solution ϵ^* , W^* , $X^* > 0$, $M^* > 0$, then the state-feedback control law is the optimal guaranteed cost controller as follows:

$$\bar{u}(\theta) = W^*(X^*)^{-1}\bar{x}(\theta) \quad (\text{A.16})$$

and the associated guaranteed cost is as follows:

$$\bar{J} \leq \text{trace}(X^{-1}) = \bar{J}^* \quad (\text{A.17})$$

where W is matrix variable, and X, M are positive definite matrix variables.

The above problem is a convex optimization problem subject to linear matrix inequalities constraints and can be solved by using LMI toolbox in MATLAB.

Appendix C

Nomenclature

Subscripts and superscript

j	The j th deputy satellite
f	Final value
$\tilde{r}$	Scaled variable
$\bar{r}$	Secular part of orbit element variables
r^t	Time-varying variables
$\dot{r}$	First derivative
$\ddot{r}$	Second derivative
v	The v th phase in the trajectory
0	Initial value
0	Reference or chief satellite

The following list of symbols is alphabetical—lowercase and then upper case; Arabic; and then Greek letters.

a	Semimajor axis
$\mathbf{a}$	Control acceleration vector
a_x	Control acceleration in x direction in LVLH frame
a_y	Control acceleration in y direction in LVLH frame
a_z	Control acceleration in z direction in LVLH frame
c_o	$\cos(o)$
d	Disturbance torque
$\bar{d}$	Distance between satellites
$\bar{d}_{\max}$	Possible maximum relative distance during the maneuver
$\bar{d}_{\text{safe}}$	Assumed safe distance between satellites
e	Eccentricity
$\mathbf{e}$	Euler axis
f	True anomaly
g_0	Sea-level acceleration due to gravity

$\mathbf{h}$	Angular momentum vector
h	Magnitude of angular momentum vector
i	Inclination
i	Index (general)
j	Index (general)
m	Satellite mass
$\mathbf{q}$	Relative position vector in LVLH frame
$\bar{q}$	Error quaternion
q	Vector part of error quaternion
q_0	Scalar part of error quaternion
$\mathbf{r}$	Position vector of reference or chief satellite
r	Geocentric distance of reference or chief satellite
s_o	$\sin(o)$
s_j	The j th component of multi-satellite sliding vector
t	General time
u	Thrust direction
u	Control torque
v_{jx}	Component of velocity vector of the j th member satellite in ECI frame
v_{jy}	Component of velocity vector of the j th member satellite in ECI frame
v_{jz}	Component of velocity vector of the j th member satellite in ECI frame
x	Radial difference between two objects in LVLH frame
$\hat{x}$	Unit vector along radial direction in LVLH frame
y	Along-track difference between two objects in LVLH frame
$\hat{y}$	Unit vector along along-track direction in LVLH frame
z	Cross-track difference between two objects in LVLH frame
$\hat{z}$	Unit vector along cross-track direction in LVLH frame
D	Reference frame for satellite reference attitude
E	Orbital energy
F_j	Magnitude of constant thrust of the j th member satellite
H	Polar component of the orbital angular momentum
I_{sp}	The specific impulse of the engine
J	Inertia matrix of satellite
J_2	Geopotential coefficient representing Earth's oblateness
K_j	Kinetic energy of the j th member satellite
R	Rotation Matrix
r_{cf}	Radius of circular formation
R_e	Earth equatorial radius
S	Multi-satellite sliding vector
S_o	Reference satellite or chief satellite
S_j	The j th member satellite
L_j	Lagrangian of the j th member satellite
L	Laplacian matrix in algebraic graph theory
M	Satellite mean anomaly
N	Number of LGL points
N_s	Number of deputy satellites

T	Nodal orbital period
T	Magnitude of constant thrust
T_c	Period of chief satellite
U	Gravitational potential
$\hat{\mathbf{X}}$	Unit vector in Earth-centered inertial (ECI) frame
$\hat{\mathbf{Y}}$	Unit vector in ECI frame
$\hat{\mathbf{Z}}$	Unit vector in ECI frame
α_x	Satellite steering acceleration
α_z	Satellite orbital acceleration
$\mathbf{p}$	Position vector in LVLH frame
ρ	Magnitude of position vector in LVLH frame
θ	True latitude
ϕ	Geocentric latitude of satellite
φ	Euler angle
$\bar{\eta}$	Quaternion
η	Vector part of quaternion
η_0	Scalar part of quaternion
ϕ	Geocentric latitude of satellite
Ω	Right ascension of ascending node
$\boldsymbol{\omega}$	Vector of orbital angular velocity
ω	Argument of perigee
ω	Angular velocity
ω_d	Desired angular velocity
$\tilde{\omega}$	Angular velocity error
ω_x	Component of orbital angular velocity
ω_y	Component of orbital angular velocity
ω_z	Component of orbital angular velocity
μ	Earth gravitational constant
λ	Eigenvalue

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